TEMA Tend. Mat. Apl. Comput., **11**, No. 1 (2010), 89-100.

Analyzing the Relationship between Interval-valued D-Implications and Interval-valued QL-Implications[1]

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Abstract. The aim of this work is to analyze the relationship between interval QL-implications and their contrapositions named interval D-implications. In order to achieve this aim, the commutative classes relating to these concepts are studied. We also analyze under which conditions the main properties corresponding to punctual D-implications and QL-implications are still valid when an interval-based fuzzy approach, on the best interval representation, is considered.

Keywords. Interval Fuzzy Logic, Interval Fuzzy Implication, D-implication, QL-implication.

1. Introduction

The interval-valued fuzzy set (IVFS) theory was introduced in an independent way by [18, 20, 31, 40] and considers the integration of Fuzzy Theory [39] and Interval Mathematics [26]. It has been studied from several viewpoints (see, e.g., [14, 17, 21, 27, 37, 16]) to deal with the representation of different kinds of uncertainty in fuzzy logic. In this paper, an interval extension considers the best interval representation of a fuzzy membership function [5, 34] to model the uncertainty in the process of determining exact membership grades.

In fuzzy logic, fuzzy implications provide a methodology for formalizing information based on expert's statements in complex systems, since a large part of expert

[1] This work was partially supported by CNPq (Proc. 473201/07-0, 307185/07-9, 476933/07-2, 502999/08-0, 308256/2009-3, 483257/2009-5).

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Recebido em 04 Setembro 2009; Aceito em 14 Junho 2010.

knowledge consists of if-then statements. Among different models of fuzzy implications, this work focuses on *quantum logic implications* (QL-implications, for short) and their contrapositions named *Dishkant implications* (D-implications, for short). Historically, in 1970, the QL-implication $x \to b = a' \vee (a \wedge b)$ was found to satisfy the condition $a \to b \Leftrightarrow a \leq b$, and reformulated an orthomodular lattice $L = (L, \vee, \wedge)$ (as the lattice of closed subspaces of Hilbert spaces) in a logic-like way, where conjunction and disjunction are non distributive. In 1974, Hermann Dishkant introduced the first-predicate logic based on the D-implication $x \to b = (b' \wedge a') \vee b$ [22].

The interrelationship of such fuzzy implications have been studied recently in [23, 24] and have been used in fuzzy rule based systems and in their inference processes. Interval automorphisms acting on D-implications were considered in [29]. In addition, a discussion under which conditions interval-valued fuzzy QL-implications preserve properties of canonical forms, generated by interval t-conorms and interval t-norms, is introduced in [28]. Thus, commutative classes of interval QL-implications and interval D-implications can be generated. It is also shown that interval QL-implications can be constructed from interval automorphisms, which are preserved by the interval canonical representation introduced in our previous works [3, 4, 5, 34].

The aim of this paper is to analyze the relationship between interval QL-implications [28] and interval D-implications [29], studying several important properties which relate these concepts. In particular, we analyze under which conditions the main properties relating punctual D-implications and QL-implications [23, 25] are still valid when an interval-based fuzzy approach is considered. In this case, the analysis of properties, as the contrapositive symmetry, the exchange principle, the first place antitonicity and the second place isotonicity, is taken into account.

This paper is organized as follows. Section 2. reviews the main concepts related to interval representations. Interval fuzzy t-conorms (t-norms) and negations are presented in sections 3. and 4., respectively. Fuzzy implications, interval QL-implications and interval D-implications are discussed in the sequence. The interrelations between interval QL-implications and interval D-Implications are considered in Section 8.. Section 9. is the Conclusion with some final remarks.

2. Interval Representations

Let $\mathbb{U} = \{[a, b] \mid 0 \leq a \leq b \leq 1\}$ be the set of subintervals of $U = [0, 1] \subseteq \mathbb{R}$. The *left* and *right projections* $l, r : \mathbb{U} \to U$ are defined by $l([a, b]) = a$ and $r([a, b]) = b$, respectively. For a given interval $X \in \mathbb{U}$, $l(X)$ and $r(X)$ are also denoted by $\underline{X}$ and $\overline{X}$, respectively.

Among the partial orders that may be defined on $\mathbb{U}$ [8], in this work we consider:
- *Product order*: $X \leq Y \Leftrightarrow \underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$ (the component-wise Kulisch-Miranker order).
- *Inclusion order*: $X \subseteq Y \Leftrightarrow \underline{X} \geq \underline{Y}$ and $\overline{X} \leq \overline{Y}$.

Definition 2.1 ([32]). *$F : \mathbb{U}^n \longrightarrow \mathbb{U}$ is an interval representation of a function $f : U^n \longrightarrow U$ if, for each $\vec{X} \in \mathbb{U}^n$ and $\vec{x} \in \vec{X}$, $f(\vec{x}) \in F(\vec{X})$.*[5]

[5]Observe that the concept of interval representation is different from interval extension and

$F : \mathbb{U}^n \longrightarrow \mathbb{U}$ is a better interval representation of $f : U^n \longrightarrow U$ than $G : \mathbb{U}^n \longrightarrow \mathbb{U}$, denoted by $G \sqsubseteq F$, if, for each $\vec{X} \in \mathbb{U}^n$, $F(\vec{X}) \subseteq G(\vec{X})$.

Definition 2.2 ([32]). *The best interval representation of a real function $f : U^n \longrightarrow U$ is the interval function $\widehat{f} : \mathbb{U}^n \longrightarrow \mathbb{U}$, defined by*

$$\widehat{f}(\vec{X}) = [\inf\{f(\vec{x}) \mid \vec{x} \in \vec{X}\}, \sup\{f(\vec{x}) \mid \vec{x} \in \vec{X}\}]. \tag{2.1}$$

The interval function $\widehat{f}$ is well defined and for any other interval representation F of f, $F \sqsubseteq \widehat{f}$. Thus, $\widehat{f}$ returns a narrower interval than any other interval representation of f and $\widehat{f}$ has the *optimality property* of interval algorithms [19], when it is seen as an algorithm to compute a real function f.

Remark 1. *The best interval representation of a function $f : U^n \longrightarrow U$ is the hull of the range of f, that is, for each $X \in \mathbb{U}^n$, $\widehat{f}(\vec{X}) = \{f(\vec{x}) : \vec{x} \in \vec{X}\} = f(\vec{X})$ if and only if f is continuous in the usual sense [32].*

Proposition 2.1. *Let $f : U^n \to U$ and $X_1, \ldots, X_n, Y_1, \ldots, Y_n \in \mathbb{U}$. If $X_i \subseteq Y_i$, for $i = 1 \ldots n$, then it holds that $\widehat{f}(X_1, \ldots, X_n) \subseteq \widehat{f}(Y_1, \ldots, Y_n)$.*

Proof. It is straightforward. □

3. Interval t-norms and t-conorms

A triangular conorm (norm), *t-conorm* (*t-norm*) for short, is a function $S(T) : U^2 \to U$ that is commutative, associative, monotonic and has 0 (1) as neutral element.

Example 1. *The maximum t-conorm and the minimum t-norm are expressed, respectively, as:*

$$S_M(x, y) = \max\{x, y\}, \quad T_M(x, y) = \min\{x, y\}.$$

In the following, we present the interval generalizations of t-conorms (t-norms).

Definition 3.1 ([5, 34]). *A function $\mathbb{S} : \mathbb{U}^2 \to \mathbb{U}$ is an interval t-conorm (t-norm) if it is commutative, associative, monotonic w.r.t. the product and inclusion order and $[0, 0]$ ($[1, 1]$) is the neutral element.*

Proposition 3.1. ([5, Theorem 4.1,Theorem 4.3] [34, Proposition 3.1]). *If S (T) is a t-conorm (t-norm) then $\widehat{S}(\widehat{T}) : \mathbb{U}^2 \to \mathbb{U}$ is an interval t-conorm (t-norm).*

Characterizations of $\widehat{S}$ and $\widehat{T}$ are given, respectively, by:

$$\widehat{S}(X, Y) = [S(\underline{X}, \underline{Y}), S(\overline{X}, \overline{Y})], \tag{3.1}$$
$$\widehat{T}(X, Y) = [T(\underline{X}, \underline{Y}), T(\overline{X}, \overline{Y})]. \tag{3.2}$$

Notice that [17, Theorem 7 and Theorem 12] provide an apparently equivalent result of Proposition 3.1. However, the notion of interval t-norm and t-cnorm used by Gehrke et al is essentially different of the notion reported in Definition 3.1.

natural extension [26].

4. Interval Fuzzy Negations

A function $N : U \to U$ is a *fuzzy negation* if

N1 : $N(0) = 1$ and $N(1) = 0$;

N2 : If $x \geq y$ then $N(x) \leq N(y)$, $\forall x, y \in U$.

Fuzzy negations satisfying the involutive property are called *strong fuzzy negations*:

N3 : $N(N(x)) = x$, $\forall x \in U$.

Definition 4.1 ([34]). *An interval function $\mathbb{N} : \mathbb{U} \longrightarrow \mathbb{U}$ is an interval fuzzy negation if, for any X, Y in $\mathbb{U}$, the following properties hold:*

$\mathbb{N}1$: $\mathbb{N}([0,0]) = [1,1]$ *and* $\mathbb{N}([1,1]) = [0,0]$*;*

$\mathbb{N}2$: *If $X \geq Y$ then $\mathbb{N}(X) \leq \mathbb{N}(Y)$;*

$\mathbb{N}3$: *If $X \subseteq Y$ then $\mathbb{N}(X) \subseteq \mathbb{N}(Y)$.*

If $\mathbb{N}$ also meets the involutive property, it is said to be a strong interval fuzzy negation:

$\mathbb{N}4$: $\mathbb{N}(\mathbb{N}(X)) = X$, $\forall X \in \mathbb{U}$.

Definitions 3.1 and 4.1 are equivalent to the definitions of t-representable t-norm, t-conorm and negation on $\mathcal{L}^I$ ($\mathbb{U}$ in this paper) which are used by the "Fuzziness and Uncertainty Modelling Research Unit", at Ghent University, see [13].

Let $N : U \longrightarrow U$ be a fuzzy negation. A characterization of $\widehat{N}$ is given by:

$$\widehat{N}(X) = [N(\overline{X}), N(\underline{X})]. \tag{4.1}$$

Proposition 4.1. ([2, Theorem 5.3])[6]. *$\mathbb{N} : \mathbb{U} \longrightarrow \mathbb{U}$ is a strong interval fuzzy negation if and only if there exists a strong fuzzy negation N such that $\mathbb{N} = \widehat{N}$.*

5. Fuzzy Implications

Several definitions for *fuzzy implications* have been given (see, e.g., [6, 15, 30, 38, 1]). The minimal properties that a fuzzy implication $I : U^2 \longrightarrow U$ must satisfy are the corner conditions:

$$I(1,1) = I(0,1) = I(0,0) = 1 \text{ and } I(1,0) = 0.$$

[6] An analogous result, in the context of Atanassov's Intuitionistic fuzzy sets (AIFS), can be found in [11, Theorem 3.6]. AIFS are extensions of fuzzy sets, introduced by Krassimir Atanassov in 1983, equivalent to IVFS in the sense that it is possible to convert AIFS into IVFS and vice-versa [10].

There are different classes of fuzzy implications obtained in a canonical way from other fuzzy connectives. This paper focuses on the classes of QL-implications and D-implications.

Let S be a t-conorm, N be a strong fuzzy negation and T be a t-norm. A QL-implication is a fuzzy implication defined, for all $x, y \in [0,1]$, by [33]:

$$I_{S,N,T}(x,y) = S(N(x), T(x,y)). \tag{5.1}$$

an a D-implication is a fuzzy implication defined, for all $x, y \in [0,1]$, by [23, 24]:

$$I_{S,T,N}(x,y) = S(T(N(x), N(y)), y). \tag{5.2}$$

The definitions of QL-implication and D-implication are closely related. In fact, a D-implication is just the contraposition with respect to N of a QL-implication, and the converse also holds. Thus, whenever $I_{S,N,T}$ is a QL-implication given by Eq.(5.1), $I_{S,T,N}$ is a D-implication given by Eq.(5.2), and N is their underlying fuzzy negation, then we have that

$$I_{S,T,N}(x,y) = I_{S,N,T}(N(y), N(x)) \quad \text{and} \quad I_{S,N,T}(x,y) = I_{S,T,N}(N(y), N(x)) \tag{5.3}$$

are a D-implication and a QL-implication, respectively [24].

Proposition 5.1. ([23, Proposition 3][35, 36]). *Let T be a t-norm, S a t-conorm and N a strong fuzzy negation. If $I_{S,N,T}$ (or $I_{S,T,N}$) satisfies the property $x \leq z \Rightarrow I(x,y) \geq I(z,y)$, then*

$$S(x, N(x)) = 1, \text{ for all } x \in U. \tag{5.4}$$

6. Interval QL-implications

An interval function $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}} : \mathbb{U}^2 \to \mathbb{U}$ is an **interval QL-implication** if there is an interval t-conorm $\mathbb{S}$, a strong interval fuzzy negation $\mathbb{N}$ and an interval t-norm $\mathbb{T}$ such that

$$\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}(X,Y) = \mathbb{S}(\mathbb{N}(X), \mathbb{T}(X,Y)). \tag{6.1}$$

Proposition 6.1. ([28, Theorem 4]). *Let S be a t-conorm, N be a fuzzy negation and T be a t-norm. If S, T and N are continuous, then*

$$\mathbb{I}_{\widehat{S},\widehat{N},\widehat{T}} = \widehat{I_{S,N,T}}. \tag{6.2}$$

Corollary 6.0. ([28, Corollary 1]). *If I is a continuous QL-Implication then $\widehat{I}$ is an interval QL-implication.*

Denote by $\mathcal{C}(S)$, $\mathcal{C}(T)$, $\mathcal{C}(I_{QL})$ and $\mathcal{C}(N)$ the classes of continuous t-conorms, t-norms and QL-implications, and strong fuzzy negations, respectively. The related interval extensions are indicated by $\mathcal{C}(\mathbb{S})$, $\mathcal{C}(\mathbb{T})$, $\mathcal{C}(\mathbb{I}_{\mathbb{QL}})$ and $\mathcal{C}(\mathbb{N})$, respectively. The results in previous sections and Proposition 6.1 state the commutativity of the diagram in Figure 1.

$$\begin{tikzcd}
\mathcal{C}(S) \times \mathcal{C}(N) \times \mathcal{C}(T) \arrow[r, "Eq.(5.1)"] \arrow[d, "(Eq.(3.1), Eq.(4.1), Eq.(3.2))"'] & \mathcal{C}(I_{QL}) \arrow[d, "Eq.(6.2)"] \\
\mathcal{C}(\mathbb{S}) \times \mathcal{C}(\mathbb{N}) \times \mathcal{C}(\mathbb{T}) \arrow[r, "Eq.(6.1)"] & \mathcal{C}(\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}})
\end{tikzcd}$$

Figure 1: Commutative classes of interval QL-implications

7. Interval D-implications

An interval function $\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}} : \mathbb{U}^2 \rightarrow \mathbb{U}$ is an *interval D-implication* whenever there exists an interval t-conorm $\mathbb{S}$, an interval t-norm $\mathbb{T}$ and a strong interval fuzzy negation $\mathbb{N}$ such that

$$\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}(X, Y) = \mathbb{S}(\mathbb{T}(\mathbb{N}(X), \mathbb{N}(Y)), Y). \tag{7.1}$$

Proposition 7.1. ([29, Proposition 7.1]). *Let S be a t-conorm, T be a t-norm and N be a fuzzy negation. If S, T and N are continuous, then*

$$\mathbb{I}_{\widehat{S},\widehat{T},\widehat{N}} = \widehat{I_{S,T,N}}. \tag{7.2}$$

Corollary 7.0. ([29, Corollary 7.2]). *If I is a continuous D-Implication then $\widehat{I}$ is an interval D-implication.*

Denote by $\mathcal{C}(S)$, $\mathcal{C}(T)$, $\mathcal{C}(I_D)$ and $\mathcal{C}(N)$ the classes of continuous t-conorms, t-norms and D-implications, and strong fuzzy negations, respectively. The related interval extensions are indicated by $\mathcal{C}(\mathbb{S})$, $\mathcal{C}(\mathbb{T})$, $\mathcal{C}(\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}})$ and $\mathcal{C}(\mathbb{N})$, respectively. The results presented in previous sections and Proposition 7.1 state the commutativity of the diagram in Figure 2.

$$\begin{tikzcd}
\mathcal{C}(S) \times \mathcal{C}(T) \times \mathcal{C}(N) \arrow[r, "Eq.(5.2)"] \arrow[d, "Eq.(3.1), Eq.(3.2), Eq.(4.1)"'] & \mathcal{C}(I_D) \arrow[d, "Eq.(7.2)"] \\
\mathcal{C}(\mathbb{S}) \times \mathcal{C}(\mathbb{T}) \times \mathcal{C}(\mathbb{N}) \arrow[r, "Eq.(7.1)"] & \mathcal{C}(\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}})
\end{tikzcd}$$

Figure 2: Commutative classes of interval D-implications

8. Relating Interval QL-implications and Interval D-Implications

The results presented in this section are based on the study of the interrelations between QL-implications and D-implications, which have appeared only recently as subject of some important works (see, e.g., [23, 24]). In this paper, we analyze important properties connecting interval D-implications and interval QL-implications.

Proposition 8.1. *Let $\mathbb{T}$ be an interval t-norm, $\mathbb{S}$ an interval t-conorm and $\mathbb{N}$ a strong interval fuzzy negation. If $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}$ (or $\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}$) satisfies the first place antitonicity property FPA: $X \leq Z \Rightarrow \mathbb{I}(X,Y) \geq \mathbb{I}(Z,Y)$ (correspondingly the second place isotonicity property SPI: $Y \leq Z \Rightarrow \mathbb{I}(X,Y) \leq \mathbb{I}(X,Z)$) then it holds that*

$$\mathbb{S}(X, \mathbb{N}(X)) = [1,1], \quad \textit{for all} \quad X \in \mathbb{U}. \tag{8.1}$$

Proof. Suppose that $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}$ satisfies the property FPA. Then, it follows that

$$\begin{aligned}
\mathbb{S}(X, \mathbb{N}(X)) &= \mathbb{S}(\mathbb{T}(X,[1,1]), \mathbb{N}(X)) \\
&= \mathbb{S}(\mathbb{N}(X), \mathbb{T}(X,[1,1])) \\
&= \mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}(X,[1,1]) \\
&\geq \mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}([1,1],[1,1]) \ \text{(by FPA)} \\
&= [1,1]
\end{aligned}$$

Thus, one has that $\mathbb{S}(X,\mathbb{N}(X)) = [1,1]$. The other case is analogous, considering SPI instead of FPA. □

Proposition 8.2. *Let $\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}$ be an interval D-implication, $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}$ an interval QL-implication and $\mathbb{N}$ their underlying interval fuzzy negation. Then, it holds that*

$$\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}(\mathbb{N}(Y), \mathbb{N}(X)) = \mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}(X,Y) \ \textit{and} \ \mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}(\mathbb{N}(Y), \mathbb{N}(X)) = \mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}(X,Y). \tag{8.2}$$

Proof. Suppose that $\mathbb{I}_{QL}$ is an interval QL-implication. Then, it follows that

$$\begin{aligned}
\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}(\mathbb{N}(Y), \mathbb{N}(X)) &= \mathbb{S}(\mathbb{N}(\mathbb{N}(Y)), \mathbb{T}(\mathbb{N}(Y), \mathbb{N}(X))) \\
&= \mathbb{S}(Y, \mathbb{T}(\mathbb{N}(X), \mathbb{N}(Y))) \\
&= \mathbb{S}(\mathbb{T}(\mathbb{N}(X), \mathbb{N}(Y)), Y) \\
&= \mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}(X,Y)
\end{aligned}$$

Therefore, $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}(\mathbb{N}(Y), \mathbb{N}(X))$ is an interval D-implication. The other case is analogous. □

Proposition 8.3. *Let T_M be the minimum t-norm presented in Example 1, $\mathbb{N}$ a strong interval fuzzy negation and $\mathbb{S}$ an interval t-conorm satisfying Eq. 8.1. Then, it holds that*

$$\mathbb{I}_{\mathbb{S},\mathbb{N},\widehat{T_M}}(X,Y) \subseteq \mathbb{I}_{\mathbb{S},\widehat{T_M},\mathbb{N}}(X,Y) = \begin{cases} [1,1] & \textit{if } X \leq Y, \\ \mathbb{S}(\mathbb{N}(X), Y) & \textit{if } X \geq Y; \\ \mathbb{S}(\mathbb{N}([\underline{X}, \overline{Y}]), Y) & \textit{if } X \subseteq Y; \\ \mathbb{S}(\mathbb{N}([\underline{Y}, \overline{X}]), X) & \textit{if } Y \subseteq X \end{cases} \tag{8.3}$$

and, for $I_{\mathbb{S},\widehat{T_M},\mathbb{N}}$, the first place antitonicity property holds.

Proof. Suppose that T_M is the minimum t-norm presented in Example 1, $\mathbb{N}$ is a strong interval fuzzy negation and $\mathbb{S}$ is an interval t-conorm satisfying Eq. 8.1. Then, it follows that
(i) If $X \leq Y$ then $\mathbb{I}_{\mathbb{S},\mathbb{N},\widehat{T_M}}(X,Y) = \mathbb{S}(\mathbb{N}(X), \widehat{T_M}(X,Y)) = \mathbb{S}(X, \mathbb{N}(X)) = [1,1]$ and $\mathbb{I}_{\mathbb{S},\widehat{T_M},\mathbb{N}}(X,Y) = \mathbb{S}(\widehat{T_M}(\mathbb{N}(X),\mathbb{N}(Y)),Y) = \mathbb{S}(\mathbb{N}(Y),Y) = [1,1]$.
(ii) If $Y \leq X$ then $\mathbb{I}_{\mathbb{S},\mathbb{N},\widehat{T_M}}(X,Y) = \mathbb{S}(\mathbb{N}(X), \widehat{T_M}(X,Y)) = \mathbb{S}(\mathbb{N}(X),Y)$ and $\mathbb{I}_{\mathbb{S},\widehat{T_M},\mathbb{N}}(X,Y) = \mathbb{S}(\widehat{T_M}(\mathbb{N}(X),\mathbb{N}(Y)),Y) = \mathbb{S}(\mathbb{N}(X),Y)$.
(iii) If $X \subseteq Y$ then

$$\begin{aligned}
\mathbb{I}_{\mathbb{S},\mathbb{N},\widehat{T_M}}(X,Y) &= \mathbb{S}(\mathbb{N}(X), \widehat{T_M}(X,Y)) \\
&= \mathbb{S}(\mathbb{N}(X), [\underline{Y}, \overline{X}]) \\
&\subseteq \mathbb{S}(\mathbb{N}([\underline{X}, \overline{Y}]), Y) \\
&= \mathbb{S}([N(\overline{Y}), N(\underline{X})], Y) \\
&= \mathbb{S}(\widehat{T_M}([N(\overline{X}), N(\underline{X})], [N(\overline{Y}), N(\underline{Y})]), Y) \\
&= \mathbb{S}(\widehat{T_M}(\widehat{N}(X), \widehat{N}(Y)), Y) \\
&= \mathbb{S}(\widehat{T_M}(\mathbb{N}(X), \mathbb{N}(Y)), Y) \\
&= I_{\mathbb{S},\widehat{T_M},\mathbb{N}}(X,Y).
\end{aligned}$$

(iv) The proof for $\mathbb{S}(\mathbb{N}([\underline{Y}, \overline{X}], X)$, if $Y \subseteq X$, is analogous to the case (iii).

The proof for the FPA property related to $I_{\mathbb{S},\widehat{T_M},\mathbb{N}}$ follows from the antitonicity of $\mathbb{N}$ and the monotonicity of $\widehat{T_M}$ and $\mathbb{S}$. □

Given any interval QL-implication (D-implication) $\mathbb{I}$, it holds that $\mathbb{I}(X,[0,0]) = \mathbb{N}(X)$. Interrelations between interval QL-implications and interval D-Implications may be analyzed with respect to other properties. For example, if an interval QL-implication or interval D-implication $\mathbb{I}$ satisfies the exchange principle, that is, $\mathbb{I}(X, \mathbb{I}(Y,Z)) = \mathbb{I}(Y, \mathbb{I}(X,Z))$, then it also satisfies the contrapositive symmetry with respect to a strong interval negation $\mathbb{N}$, that is:

$$\mathbb{I}(\mathbb{N}(X), \mathbb{N}(Y)) = \mathbb{I}(\mathbb{N}(Y), \mathbb{I}(X,[0,0])) = \mathbb{I}(X, \mathbb{I}(\mathbb{N}(Y),[0,0])) = \mathbb{I}(X,Y).$$

Proposition 8.4. *Let $\mathbb{T}$ be an interval t-norm, $\mathbb{S}$ an interval t-conorm and $\mathbb{N}$ as an interval strong negation such that the corresponding interval QL-implication $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}$ (equivalently, the interval D-implication $\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}$) is an implication satisfying the property FPA (SPI). The following statements are equivalent:*

1. *$\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}$ satisfies the exchange principle;*
2. *$\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}$ satisfies the exchange principle;*
3. *$\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}$ is an interval S-implication[7];*

[7]The notion of interval S-implication considered here is the same considered in [3], i.e. interval functions $\mathbb{I}$ such that $\mathbb{I}(X,Y) = \mathbb{S}(\mathbb{N}(X),Y)$ for some interval t-conorm $\mathbb{S}$ and interval fuzzy negation $\mathbb{N}$.

4. *$\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}$ is an interval S-implication;*

5. *There exists a t-conorm S_1 such that*

$$\mathbb{S}(\mathbb{N}(X), \mathbb{T}(X,Y)) = \mathbb{S}_1(\mathbb{N}(X), Y). \tag{8.4}$$

Proof. Only the equivalence among 1, 3 and 5 will be proved, since it is possible to prove the equivalence among 2, 4 and 5 following an analogous reasoning.

- ($1 \Rightarrow 3$) If $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}$ satisfies the exchange principle, then it also satisfies contrapositive symmetry and then, by [3, Theorem 29], $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}$ is an interval S-implication.

- ($3 \Rightarrow 5$) If $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}$ is an interval S-implication, then there exists an interval t-conorm $\mathbb{S}_1$ and an interval strong negation $\mathbb{N}_1$ such that $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}(X,Y) = \mathbb{S}(\mathbb{N}(X), \mathbb{T}(X,Y)) = \mathbb{S}_1(\mathbb{N}_1(X), Y)$ for all $X, Y \in \mathbb{U}$. But taking $Y = [0,0]$ in the above equation, we obtain $\mathbb{N}(X) = \mathbb{N}_1(X)$, for all $X \in \mathbb{U}$, and consequently Eq.(8.4) follows.

- ($5 \Rightarrow 1$) If $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}$ satisfies the Eq (8.4), then $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}$ is an interval S-implication and so, by [3, Theorem 29], it satisfies the exchange principle.

□

9. Conclusion and Final Remarks

In this paper, the notion of interval t-norm (t-conorm and negation) is based on the previous work of the authors [5, 34, 2, 3], where interval fuzzy connectives are seen as interval representations of fuzzy connectives, obtained by the canonical representation. In such context, the interval t-norm notion adds the property of inclusion monotonicity to the notions of interval t-norms considered in [11, 9, 16, 7, 12]. As a consequence, each interval t-norm (t-conorm and negation) in the sense of Santiago et al [32] is also representable in the sense of Deschrijver et al [9]. In fact, the representable notions proposed in [9] and the canonical representation proposed in [32] are distinct notions. Whereas the first one associates an interval t-norm to two t-norms (their related representant) the second one associates an interval t-norm to a t-norm, which is not necessarily a representant.

Among the usual fuzzy implications, which are used in fuzzy rule based systems and in performing inferences in approximate reasoning and fuzzy control, this paper focused on two classes of interval valued implications: interval-valued QL-implications and interval-valued D-implications. The latter ones are contrapositions of the former ones, with respect to strong interval fuzzy negations, and vice-versa. Both interval-valued implications are constructed from two special aggregation operators: interval t-norms and interval t-conorms. The paper mainly discussed under which conditions generalized fuzzy QL-implications and D-implications, applied to interval values, preserve properties of canonical forms. It was shown that some properties of fuzzy logic (e.g, contrapositive symmetry, exchange principle, first place antitonicity, second place isotonicity) might be naturally extended to interval fuzzy degrees, considering the respective degenerated intervals. The study of the relationships between interval-valued D-Implications and interval-valued QL-Implications obtained from the action of an interval automorphism is an ongoing work.

Notice that, there are other works on interval-valued fuzzy implications and Atanassov's intuitionistic fuzzy implications. However, we did not find any work, except [28, 29], which extend QL-implications and D-implications for interval or intuitionistic framework.

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