LINEAR STATIC AND DYNAMIC ANALYSIS OF THIN LAMINATED COMPOSITE STRUCTURES WITH A TRIANGULAR FINITE ELEMENT

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ABSTRACT
Linear static and dynamic behavior of thin laminate composite structures are analyzed in this study using the Finite Element Method (FEM). Triangular elements with three nodes and six degrees of freedom per node (three displacement and three rotation components) are used. For static analysis the equilibrium equations are solved using Pre-conditioned Gradient Conjugate Method (GCM) while the dynamic solution is performed using the classical Newmark Method. Analytical evaluation of consistent element mass matrix and determination of membrane and membrane-bending coupling element stiffness matrix in the explicit form are showed. Numerical examples are presented and compared with results obtained by other authors with different types of elements and different schemes, proving the validity and effectiveness of the developed model.

Key-words: Laminate composite. Static and dynamic analysis. Consistent mass matrix. Finite element method.

1. INTRODUCTION

It is well known that laminate composite materials are nowadays commonly used in aeronautical, aerospace, naval and other industries mainly because of their attractive properties as compared to isotropic materials, such as higher stiffness/weight, higher strength, higher damping and good properties related to thermal or acoustic isolation, among others.

A triangular finite element called GPL-T9 presented previously by Zhang, Lu and Kuang (1998) and by Teixeira (2001) for isotropic materials was extended to laminated composite materials.

It was considered the Classical Lamination Theory (CLT) given by Jones (1999), where the complete laminate, having several layers, is analyzed as an equivalent material with only one layer.

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Two important original contributions were developed in this research: the analytical evaluation of consistent element mass matrix and the determination of membrane and membrane-bending coupling element stiffness matrix in the explicit form.

For static analysis the equilibrium equations are solved using the Gradient Cojugate Method (GCM) (see CASTRO, 1997), with diagonal preconditioning, while the dynamic solution is performed using the classical Newmark Method (see BATHE, 1996).

Examples are analyzed and compared with results obtained by other authors, showing that this element, where its mass and stiffness matrices can be implemented analytically, is able to solve structures involving thin plates and shells of composite materials efficiently.

2. TRIANGULAR FINITE ELEMENT FOR THIN LAMINATED STRUCTURES

The FIGURE 1 shows the finite element used in this work. It is a triangular element, called GPL-T9 (see ZHANG, LU AND KUANG, 1998), with three nodes and six degrees of freedom per node (three displacement and three rotation components). This is a conforming element with the compatibility conditions being satisfied in each node and in each element side (see TEIXEIRA, 2001). Using the drilling degree of freedom, numerical accuracy is improved and singularity of the stiffness matrix is avoided for coplanar elements. The total stiffness matrix for each element is obtained by superposition of the membrane and bending matrices (see ISOLDI, 2008). It was considered the Classical Lamination Theory (CLT) given by Jones (1999), where the complete laminate, having several layers, is analyzed as an equivalent material with only one layer.

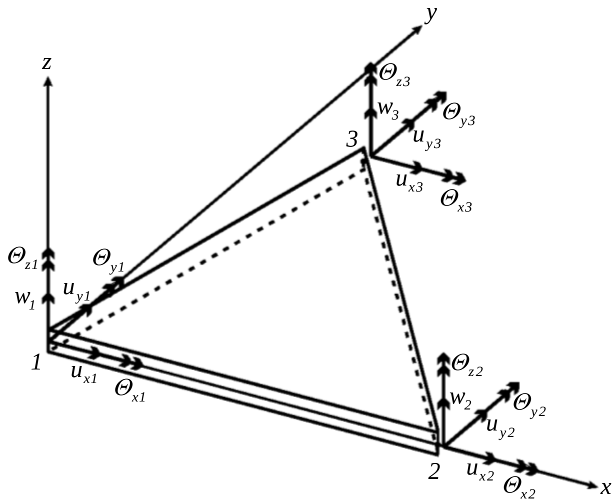

Figure 1 – Triangular element GPL-T9.

The membrane $\{u_m\}$ and bending $\{u_b\}$ displacement vectors are given by:

$$\{u_{mi}\} = \{u_{xi} \quad u_{yi} \quad \Theta_{zi}\}^T \qquad (i = 1,2,3) \tag{1}$$

$$\{u_{bi}\} = \{w_i \quad \Theta_{xi} \quad \Theta_{yi}\}^T \qquad (i = 1,2,3) \tag{2}$$

being u_{xi}, u_{yi} and w_i the displacement components and Θ_{xi}, Θ_{yi} and Θ_{zi} the rotation components.

Therefore, the equilibrium equations for dynamic analysis without damping effects, based on the Finite Element Method, for each finite element are given by:

$$[M]\{\ddot{u}\} + [K]\{u\} = \{R\} \tag{3}$$

where $[M]$ is the mass matrix, $[K]$ is the stiffness matrix, $\{R\}$ is the vector of external loads, and $\{u\}$ is the vector of displacements.

2.1 Mass matrix

In Eq. (3), the consistent mass matrix $[M]$ is defined by:

$$[M] = \sum_{k=1}^{n} h_k \rho_k \int_A [H]^T [H] dA \tag{4}$$

being n the number of layers, h_k and ρ_k the thickness and the specific mass of the layer k, respectively, A the element area, T the transpose of matrix, and $[H]$ (that represents the same shape functions used to generate the element stiffness matrix) is given by:

$$[H]=\begin{bmatrix}[H_{mi}] & [0]\\ \{0\} & \{H_{bi}\}\end{bmatrix} \tag{5}$$

being, respectively, the membrane and bending shape functions, given by

$$[H_{mi}]=\begin{bmatrix}L_i & 0 & H_{u\Theta_i}\\ 0 & L_i & H_{v\Theta_i}\end{bmatrix} \quad (i=1,2,3) \tag{6}$$

$$\{H_{bi}\}=\{H_i \quad H_{xi} \quad H_{yi}\} \quad (i=1,2,3) \tag{7}$$

with L_i representing the area coordinates, and:

$$H_{u\Theta_i}=\frac{1}{2}L_i\left(b_m L_j-b_j L_m\right) \tag{8}$$

$$H_{v\Theta_i}=\frac{1}{2}L_i\left(c_m L_j-c_j L_m\right) \tag{9}$$

$$H_i=L_i-2F_i+\left(1-r_j\right)F_j+\left(1+r_m\right)F_m \tag{10}$$

$$H_{xi}=-\frac{1}{2}\left[b_m L_i L_j-b_j L_m L_i+\left(b_j-b_m\right)F_i+\left(r_j b_j+b_m\right)F_j+\left(r_m b_m-b_j\right)F_m\right] \tag{11}$$

$$H_{yi}=-\frac{1}{2}\left[c_m L_i L_j-c_j L_m L_i+\left(c_j-c_m\right)F_i+\left(r_j c_j+c_m\right)F_j+\left(r_m c_m-c_j\right)F_m\right] \tag{12}$$

where

$$b_i=y_j-y_m \tag{13}$$

$$c_i=x_m-x_j \tag{14}$$

$$F_i=L_i\left(L_i-\frac{1}{2}\right)\left(L_i-1\right) \qquad (i,j,m=1,2,3) \tag{15}$$

$$r_i=\frac{1}{l_{j-m}^2}\left(l_{i-m}^2-l_{i-j}^2\right) \qquad (i,j,m=1,2,3) \tag{16}$$

$$l_{i-j}=\left(x_{i-j}^2+y_{i-j}^2\right)^{\frac{1}{2}} \tag{17}$$

$$x_{i-j}=x_i-x_j \tag{18}$$

$$y_{i-j}=y_i-y_j \tag{19}$$

being x_i and y_i the nodal coordinates of the element.

The analytical evaluation of consistent mass matrix of GPL-T9 element is an original aspect of this research. The mass matrix coefficients are shown In Appendix A.

2.2 Stiffness matrix

The stiffness matrix $\left[K\right]$, in Eq. (3), is formed taking into account membrane ($\left[K_m\right]$), bending ($\left[K_b\right]$) and membrane-bending coupling effects ($\left[K_{bm}\right]=\left[K_{mb}\right]^T$), that are given by:

$$\left[K_m\right]=\int_{t_A}\left[B_m\right]^T\left[D_m\right]\left[B_m\right]{}^t dA \tag{20}$$

$$\left[K_b\right]=\int_{t_A}\left[B_b\right]^T\left[D_b\right]\left[B_b\right]{}^t dA \tag{21}$$

$$\left[K_{bm}\right]=\int_{t_A}\left[B_b\right]^T\left[D_{bm}\right]\left[B_m\right]{}^t dA \tag{22}$$

$$\left[K_{mb}\right]=\int_{t_A}\left[B_m\right]^T\left[D_{mb}\right]\left[B_b\right]{}^t dA \tag{23}$$

being the membrane and bending strain-displacement matrices, respectively:

$$B_{m_i}=\frac{1}{4A}\begin{bmatrix} 2b_i & 0 & b_i\left(b_m L_j-b_j L_m\right) \\ 0 & 2c_i & c_i\left(c_m L_j-c_j L_m\right) \\ 2c_i & 2b_i & \left(c_i b_m+b_i c_m\right)L_j-\left(c_i b_j+b_i c_j\right)L_m \end{bmatrix} \quad \left(i,j,m=1,2,3\right) \tag{24}$$

$$B_{b_i}=\begin{bmatrix} H_{i,xx} & H_{xi,xx} & H_{yi,xx} \\ H_{i,yy} & H_{xi,yy} & H_{yi,yy} \\ 2H_{i,xy} & 2H_{xi,xy} & 2H_{yi,xy} \end{bmatrix} \quad \left(i,j,m=1,2,3\right) \tag{25}$$

and $\left[D_m\right]$, $\left[D_b\right]$ and $\left[D_{mb}\right]=\left[D_{bm}\right]$ are, respectively, the constitutive matrices for the membrane, bending and membrane-bending coupling effects, defined by:

$$\left[D_m\right]=\left[T_\gamma\right]^T\left[A\right]\left[T_\gamma\right] \tag{26}$$

$$\left[D_{mb}\right]=\left[D_{bm}\right]=\left[T_\gamma\right]^T\left[B\right]\left[T_\gamma\right] \tag{27}$$

$$\left[D_b\right]=\left[T_\gamma\right]^T\left[D\right]\left[T_\gamma\right] \tag{28}$$

with $\left[T_\gamma\right]$ being the rotation matrix from the global to the local coordinate system, which is defined by:

$$\left[T_{\gamma}\right]=\begin{bmatrix}\cos^2\gamma & sin^2\gamma & sin\,\gamma\cos\gamma\\ sin^2\gamma & \cos^2\gamma & -sin\,\gamma\cos\gamma\\ -2sin\,\gamma\cos\gamma & 2sin\,\gamma\cos\gamma & \cos^2\gamma-sin^2\gamma\end{bmatrix} \quad (29)$$

In Eq. (29), γ is the angle formed by the global axis x_g and the local axis x_l, as indicated in FIGURE 2, where the fibers reference system is also shown.

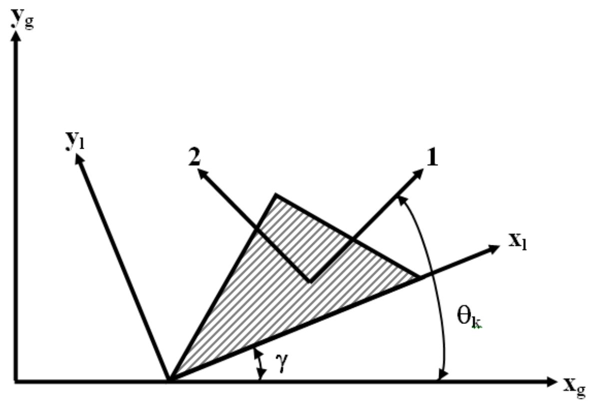

Figure 2 - Global, local and fiber coordinate systems.

In Eq. (26-28) the components of constitutive matrices are given by (JONES, 1999):

$$A_{ij}=\sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k\left(z_k-z_{k-1}\right) \qquad \left(i,j=1,2,6\right) \quad (30)$$

$$B_{ij}=\frac{1}{2}\sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k\left(z_k^2-z_{k-1}^2\right) \qquad \left(i,j=1,2,6\right) \quad (31)$$

$$D_{ij}=\frac{1}{3}\sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k\left(z_k^3-z_{k-1}^3\right) \qquad \left(i,j=1,2,6\right) \quad (32)$$

where z_{k-1} and z_k are the coordinates normal to the lower and upper surfaces of layer *k*; $\bar{Q}_{ij}$ are elastic constants of each layer *k* in the global coordinate system (see FIGURE 2) defined by the following expressions:

$$\bar{Q}_{11}=Q_{11}\cos^4\theta+2\left(Q_{12}+2Q_{66}\right)sin^2\theta\cos^2\theta+Q_{22}sin^4\theta \quad (33)$$

$$\bar{Q}_{12}=\left(Q_{11}+Q_{22}-4Q_{66}\right)sin^2\theta\cos^2\theta+Q_{12}\left(sin^4\theta+\cos^4\theta\right) \quad (34)$$

$$\bar{Q}_{16}=\left(Q_{11}-Q_{12}-2Q_{66}\right)sin\,\theta\cos^3\theta+\left(Q_{12}-Q_{22}+2Q_{66}\right)sin^3\theta\cos\theta \quad (35)$$

$$\bar{Q}_{22}=Q_{11}sin^4\theta+2\left(Q_{12}+2Q_{66}\right)sin^2\theta\cos^2\theta+Q_{22}\cos^4\theta \quad (36)$$

$$\bar{Q}_{26} = \left(Q_{11} - Q_{12} - 2Q_{66}\right) sin^3\theta \cos\theta + \left(Q_{12} - Q_{22} + 2Q_{66}\right) sin\,\theta \cos^3\theta \quad (37)$$

$$\bar{Q}_{66} = \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right) sin^2\theta \cos^2\theta + Q_{66}\left(sin^4\theta + \cos^4\theta\right) \quad (38)$$

being θ_k the angle formed by the global axis x_g and the fiber local axis *1* (see FIGURE 2), and Q_{ij} are elastic constants in the layer *k* in the fiber coordinates system and are defined by the following expressions:

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}} \quad (39)$$

$$Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \quad (40)$$

$$Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} \quad (41)$$

$$Q_{66} = G_{12} \quad (42)$$

where, for each layer *k*, E_1 and E_2 are the elastic moduli in the direction of the axis *1* and axis *2*, respectively; G_{12} is the shear modulus in the plane *1-2* in the fiber coordinate system; ν_{ij} is the Poisson coefficient defined as the relation between the strain in the transversal direction *j* and the axial strain in the direction *i*, considering the fiber coordinate system.

Another important contribution of this research is the determination of the membrane and membrane-bending coupling element stiffness matrix in the explicit form, as show in Appendix B.

2.3 External loads

The vector $\{R\}$, in Eq. (3), is composed by the nodal vectors of external loads referred to membrane and bending effects, respectively, as described by the following expressions:

$$\{R_m\} = \int_A \left[H_m\right]^T \begin{Bmatrix} \{R_x\} \\ \{R_y\} \end{Bmatrix} dA \quad (43)$$

$$\{R_b\} = \int_A \{H_b\}^T \{R_z\} dA \quad (44)$$

where $\{R_x\}$, $\{R_y\}$ and $\{R_z\}$ are the nodal vectors of external loads in *x*, *y* and *z* directions, respectively.

3. SOLUTION OF EQUILIBRIUM EQUATIONS

The equilibrium equations, previously presented, are referred to each element local coordinates, and a transformation to a common global system is necessary in order to perform the assemblage procedure. Therefore, this can be made as shown in Isoldi et al. (2008).

After that, for static analysis the equilibrium equations are solved using the Gradient Conjugate Method (GCM) (see CASTRO, 1997), with diagonal preconditioning, while the dynamic solution is performed using the classical Newmark Method (see BATHE, 1996).

4. NUMERICAL APPLICATIONS

To demonstrate the validity and effectiveness of the developed model, numerical examples are presented and compared with results obtained by other authors.

4.1 Static analysis of a clamped laminated plate under uniform loading

A clamped square laminated plate, with a stacking sequence $[0/90/90/0]$, under uniform pressure is shown in FIGURE 3. Its geometrical properties are: $a\,(\text{length}) = 308.80\times10^{-3}\,m$ and $h\,(\text{total thickness}) = 2.44x10^{-3}\,m$. Its material properties are: $E_1 = 12.60\times10^9\,Pa$, $E_2 = 12.63\times10^9\,Pa$, $G_{12} = 2.15\times10^9\,Pa$ and $\nu_{12} = 0.23949$. The uniform loading applied is $q = 13.80\times10^3\,Pa$. Owing to symmetry, only one quarter of the structure was modeled with 200 triangular elements (generated in $10x10 = 100$ rectangular regions). The boundary conditions are: $u_y = \Theta_x = \Theta_z = 0.00$ on the line $\overline{AB}$, $u_x = u_y = w = \Theta_x = \Theta_y = \Theta_z = 0.00$ on the lines $\overline{BC}$ and $\overline{CD}$ and $u_x = \Theta_y = \Theta_z = 0.00$ on the line $\overline{DA}$.

In FIGURE 4 the results were compared with those obtained by Liao and Reddy (1987) using four nine-nodes shell elements and a good agreement was obtained.

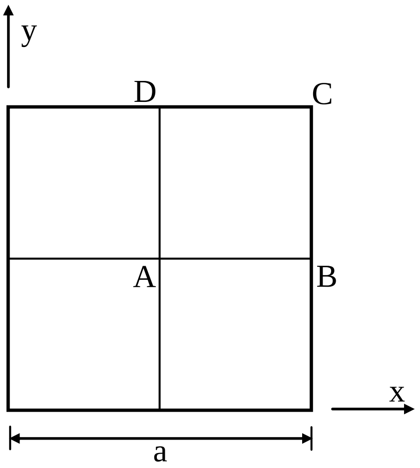

Figure 3 - Clamped square laminated plate.

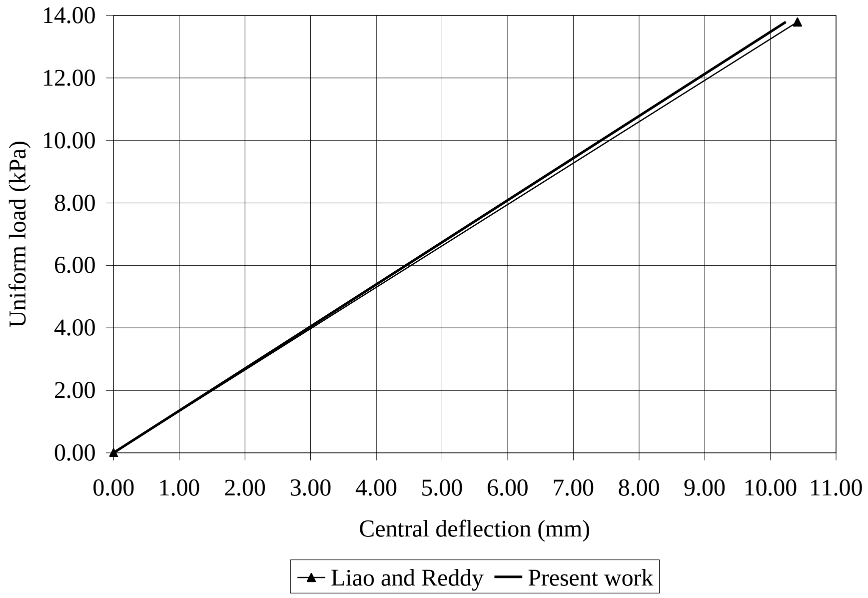

Figure 4 - four-layer [0/90/90/0] clamped plate under uniform loading.

4.2 Dynamic analysis of a cantilever beam under uniformly distributed load

In this example the behavior of a cantilever isotropic beam is analyzed. Its geometrical properties are: $L\,(\text{length}) = 254.00 \times 10^{-3}\,m$, $b\,(\text{width}) = 25.40x10^{-3}\,m$ and $h\,(\text{thickness}) = 25.40x10^{-3}\,m$. Its material properties are: $E = 82.74 \times 10^{6}\,Pa$, $G = 34.47 \times 10^{6}\,Pa$, $\nu = 0.20$ and $\rho = 10.69\,kg/m^{3}$. The uniformly distributed load is $q = 499.11 N/m$. It is considered that the structure has one end clamped

($u_x = u_y = w = \Theta_x = \Theta_y = \Theta_z = 0.00$). The beam is modeled with 160 triangular elements (generated in $(bxL)4x20 = 80$ rectangular regions). The adopted time step is $Dt = 1.35 \times 10^{-4} s$. The results are compared with those obtained by Reddy and Chandrashekhara (1985), who used a $2x2$ mesh of nine-node quadratic elements in the half width of the beam. FIGURE 5 contain plots of the transverse deflection (*w*/*L*) of the tip versus time (*t*/*Dt*).

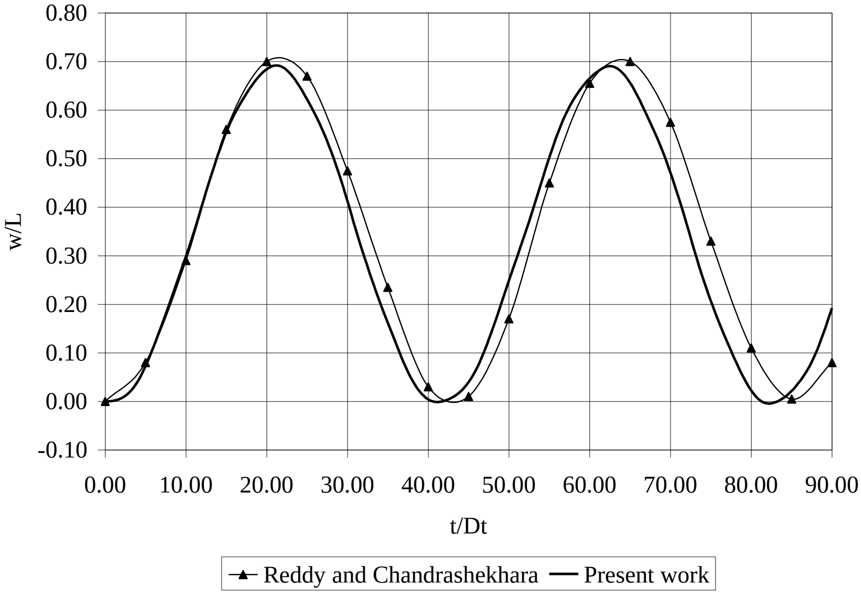

Figure 5 - Transient response of an isotropic cantilever beam under uniform.

5. FINAL REMARKS

Composite laminates are one of the most advanced structural materials nowadays, therefore it is very important to know the behavior of these structures. Therefore, the triangular finite element named GPL-T9 presented previously by Zhang, Lu and Kuang (1998) and by Teixeira (2001) for isotropic materials was extended to thin laminate structural analyses in this study.

The original aspects of this research are the analytical evaluation of consistent element mass matrix (Appendix A) and the determination of the membrane and membrane-bending coupling element stiffness matrix in the explicit form (Appendix B).

Results of linear static and dynamic examples show a very good agreement with those presented by other authors using different types of elements and formulations, demonstrating the validity and effectiveness of the developed model.

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APPENDIX A

Considering Eq. (4), it is possible to define:

$$\int_{A^e} [H]^T [H] dA^e = [\mathbf{H}]$$

and employing the integration formula (Cook, Malkus and Plesha, 1989):

$$\int_{A^e} L_1^k L_2^l L_3^m dA^e = 2A^e \frac{k!\,l!\,m!}{(2+k+l+m)!}$$

the non-zero mass matrix coefficients are given by:

$$\mathbf{H}(1,1) = \frac{A}{6};\ \mathbf{H}(1,3) = \frac{(b_3\text{-}b_2)A}{60};\ \mathbf{H}(1,7) = \frac{A}{12};\ \mathbf{H}(1,9) = \frac{A}{60}\left(\frac{1}{2}b_1 - b_3\right);\ \mathbf{H}(1,13) = \frac{A}{12};$$

$$\mathbf{H}(1,15) = \frac{(b_2 - 0.5b_1)A}{60};$$

$$\mathbf{H}(2,2) = \frac{A}{6};\ \mathbf{H}(2,3) = \frac{(c_3\text{-}c_2)A}{60};\ \mathbf{H}(2,8) = \frac{A}{12};\ \mathbf{H}(2,9) = \frac{(0.5c_1 - c_3)A}{60};\ \mathbf{H}(2,14) = \frac{A}{12};$$

$$\mathbf{H}(2,15) = \frac{(c_2 - 0.5c_1)A}{60};$$

$$\mathbf{H}(3,3) = \frac{(b_3^2\text{-}b_2b_3\text{+}b_2^2\text{+}c_3^2\text{-}c_3c_2\text{+}c_2^2)A}{360};\ \mathbf{H}(3,7) = \frac{(b_3 - 0.5b_2)A}{60};$$

$$\mathbf{H}(3,8) = \frac{(c_3 - 0.5c_2)A}{60};\ \mathbf{H}(3,9) = \frac{(b_1b_3\text{-}2b_3^2\text{-}b_1b_2\text{+}b_2b_3\text{+}c_1c_3\text{-}2c_3^2\text{-}c_1c_2\text{+}c_3c_2)A}{720};$$

$$\mathbf{H}(3,13) = \frac{(0.5b_3 - b_2)A}{60};\ \mathbf{H}(3,14) = \frac{(0.5c_3 - c_2)A}{60};$$

$$\mathbf{H}(3,15) = \frac{(b_2b_3\text{-}b_1b_3\text{-}2b_2^2\text{+}b_1b_2\text{+}c_3c_2\text{-}c_1c_3\text{-}2c_2^2\text{+}c_1c_2)A}{720};$$

$$\mathbf{H}(4,4) = \left(\frac{69213r_3^2}{58138921} - \frac{3809523r_2}{199999958} + \frac{135281r_2^2}{113636041} + \frac{3463203r_3}{181818157} + \frac{4142847}{19999951}\right)A;$$

$$\mathbf{H}(4,5)=\left(\frac{10054r_2^2b_2}{16890721}-\frac{r_3b_2r_2}{1\times10^{11}}-\frac{397500b_3}{17123077}-\frac{676406r_2b_2}{113636209}+\frac{35014r_3b_2}{29411761}-\frac{732330r_3b_3}{123031441}+\frac{268683b_2}{11574037}+\frac{b_3r_3r_2}{1.25\times10^{11}}+\frac{23434b_3r_2}{19684561}-\frac{74404r_3^2b_3}{124998719}\right)A;$$

$$\mathbf{H}(4,6)=\left(\frac{10054r_2^2c_2}{16890721}+\frac{30493r_3c_2}{25614121}+\frac{46502c_3r_2}{39061681}-\frac{14038r_3^2c_3}{23583841}\frac{676406\left(r_2c_2+r_3c_3\right)}{113636209}+\frac{362712\left(c_2-c_3\right)}{15624517}+\frac{\left(r_3c_3r_2-r_3c_2r_2\right)}{1\times10^{11}}\right)A;$$

$$\mathbf{H}(4,10)=\left(\frac{599715}{9504917}-\frac{148809r_3^2}{124999559}+\frac{r_3r_2}{1\times10^{12}}+\frac{\left(r_1r_2-r_1r_3\right)}{333333333333}+\frac{297619\left(r_2-r_1\right)}{124999979}\right)A;$$

$$\mathbf{H}(4,11)=\left(\frac{74397b_3}{6578261}-\frac{1373626b_1}{192307639}-\frac{b_3r_2}{2.5\times10^{10}}+\frac{15664r_1b_1}{13157761}-\frac{23434b_1r_3}{19684561}-\frac{248015r_3b_3}{69444201}-\frac{74404r_3^2b_3}{124998719}+\frac{b_3r_3r_2}{1\times10^{11}}-\frac{b_1r_1r_2}{333333333333}-\frac{b_1r_1r_3}{142857142857}\right)A;$$

$$\mathbf{H}(4,12)=\left(\frac{c_1r_1r_2}{166666666666}-\frac{5431r_3^2c_3}{9124081}-\frac{965250c_1}{135135001}-\frac{r_3c_3}{280}+\frac{74397c_3}{6578261}+\frac{15664r_1c_1}{13157761}-\frac{135281c_1r_3}{113636041}+\frac{r_3c_3r_2}{5\times10^{11}}-\frac{c_1r_1r_3}{1\times10^{11}}+\frac{\left(c_1r_2-c_3r_2\right)}{2.5\times10^{10}}\right)A;$$

$$\mathbf{H}(4,16)=\left(\frac{53}{840}-\frac{r_2}{333333333333}-\frac{69213r_2^2}{58138921}+\frac{r_1r_3}{1\times10^{11}}+\frac{176890r_1}{74293799}-\frac{297619r_3}{124999979}\right)A;$$

$$\mathbf{H}(4,17)=\left(-\frac{51578b_2}{4560581}+\frac{1190476b_1}{166666641}+\frac{b_1r_3}{14285714285}-\frac{1116071r_2b_2}{312499881}+\frac{67640r_2^2b_2}{113635201}-\frac{135281b_1r_2}{113636041}+\frac{14105r_1b_1}{11848201}-\frac{b_1r_1r_3}{1\times10^{11}}\right)A;$$

$$\mathbf{H}(4,18)=\left(\frac{10054r_2^2c_2}{16890721}-\frac{1116071r_2c_2}{312499881}-\frac{35014c_1r_2}{29411761}+\frac{1190476c_1}{166666641}-\frac{66987c_2}{5923061}+\frac{15664r_1c_1}{13157761}-\frac{r_3c_2r_2}{1\times10^{11}}+\frac{c_1r_1r_2}{77519379844}\right)A;$$

$$\mathbf{H}(5,5)=\left(\frac{b_3^2r_3}{840}-\frac{20807b_3b_2}{4559447}+\frac{41869b_3^2}{12412927}-\frac{r_2b_2^2}{840}-\frac{29851r_3b_3b_2}{25074841}+\frac{r_3b_3r_2b_2}{166666666666}+\frac{39404b_2^2}{11682127}+\frac{28077b_3r_2b_2}{23584681}+\frac{7154\left(r_3^2b_3^2+r_2^2b_2^2\right)}{24037439}\right)A;$$

$$\mathbf{H}(5,6)=\left(\frac{930059b_3r_3c_3}{781249559}-\frac{402187c_2b_2r_2}{337837079}-\frac{8008b_3c_2}{3509593}-\frac{b_3r_3c_2r_2}{5\times10^{11}}-\frac{r_3c_3b_2r_2}{6.25\times10^{11}}+\right.$$
$$\frac{24001b_3c_2r_2}{40321679}-\frac{21137b_3r_3c_2}{35510159}-\frac{23c_3b_2}{10080}+\frac{24001(c_3b_2r_2-r_3c_3b_2)}{40321679}+$$
$$\left.\frac{39404(c_2b_2+b_3c_3)}{11682127}+\frac{7154(r_2^2b_2c_2+r_3^2b_3c_3)}{24037439}\right)A \quad ;$$

$$\mathbf{H}(5,10)=\left(\frac{14038r_3^2b_3}{23583841}+\frac{2232142b_2}{312499881}-\frac{51578b_3}{4560581}+\frac{r_1b_2}{14285714285}-\frac{r_1b_3}{2.5\times10^{10}}-\right.$$
$$\left.\frac{248015r_3b_3}{69444201}-\frac{r_1b_3r_3}{1\times10^{11}}-\frac{r_1b_2r_2}{90909090909}+\frac{15664(r_2b_2-r_3b_2)}{13157761}\right)A \quad ;$$

$$\mathbf{H}(5,11)=\left(\frac{7154r_3^2b_3^2}{24037439}+\frac{b_1r_1b_2}{71428571428}+\frac{5079b_1b_3}{4654211}+\frac{9600b_3b_2}{8797091}-\frac{34523b_3^2}{15817811}+\right.$$
$$\frac{b_3b_2r_2}{73529411764}-\frac{4975b_1b_2}{4558909}-\frac{b_1r_1b_3}{1\times10^{11}}-\frac{b_1b_2r_2}{5\times10^{10}}-\frac{24001b_3r_3b_2}{40321679}+\frac{21137b_1b_3r_3}{35510159}- ;$$
$$\left.\frac{(b_1r_1b_3r_3+b_3r_3b_2r_2+b_1r_1b_2r_2)}{5\times10^{11}}\right)A$$

$$\mathbf{H}(5,12)=\left(\frac{25745c_1b_3r_3}{43251599}+\frac{4452c_3b_2}{4079651}-\frac{34523b_3c_3}{15817811}-\frac{13995c_1b_2}{12824509}-\frac{b_3r_3c_3}{1\times10^{12}}+\frac{9600c_1b_3}{8797091}-\right.$$
$$\frac{18884r_3c_3b_2}{31725119}-\frac{c_1b_2r_2}{76923076923}+\frac{c_3b_2r_2}{76923076923}+\frac{7154r_3^2b_3c_3}{24037439}-\frac{c_1r_1b_3}{71428571428}- \quad ;$$
$$\left.\frac{c_1r_1b_2r_2}{2.5\times10^{11}}+\frac{c_1r_1b_2}{111111111111}-\frac{(c_1r_1b_3r_3+c_3r_3b_2r_2)}{5\times10^{11}}\right)A$$

$$\mathbf{H}(5,16)=\left(\frac{74397b_2}{6578261}-\frac{b_3r_3r_2}{1\times10^{11}}-\frac{10054r_2^2b_2}{16890721}-\frac{3968253b_3}{555555421}-\frac{r_1b_2}{2.5\times10^{10}}-\frac{248015r_2b_2}{69444201}+\right.$$
$$\left.\frac{r_1b_3}{25641025641}+\frac{15664r_3b_3}{13157761}-\frac{105165b_3r_2}{88338601}+\frac{r_1b_2r_2}{111111111111}\right)A \quad ;$$

$$\mathbf{H}(5,17)=\left(\frac{5079b_3b_2}{4654211}+\frac{7782r_2^2b_2^2}{26147519}-\frac{12400b_1b_3}{11362909}+\frac{5079b_1b_2}{4654211}-\frac{34523b_2^2}{15817811}+\frac{24001b_3b_2r_2}{40321679}-\right.$$
$$\frac{b_1r_1b_3}{71428571428}-\frac{14308b_1b_2r_2}{24037439}-\frac{b_3r_3b_2}{76923076923}-\frac{(b_1r_1b_2r_2+b_3r_3b_2r_2+b_1r_1b_3r_3)}{5\times10^{11}}+$$
$$\left.\frac{(b_1b_2r_1-b_2^2r_2+b_1b_3r_3)}{1\times10^{11}}\right)A$$

;

$$\mathbf{H}(5,18)=\left(\frac{24001b_3c_2r_2}{40321679}-\frac{24001c_1b_2r_2}{40321679}+\frac{7782r_2^{\,2}b_2c_2}{26147519}-\frac{19211c_2b_2}{8802131}+\frac{85247c_1b_2}{78117251}-\right.$$

$$\frac{5778c_1b_3}{5294749}+\frac{c_2b_2r_2}{2\times10^{11}}+\frac{3825b_3c_2}{3505091}+\frac{c_1b_3r_3}{1.25\times10^{11}}+\frac{\left(c_1r_1b_2-c_1r_1b_3-b_3r_3c_2\right)}{1\times10^{11}}- \quad ;$$

$$\left.\frac{\left(c_1r_1b_2r_2+c_1r_1b_3r_3+b_3r_3c_2r_2\right)}{5\times10^{11}}\right)A$$

$$\mathbf{H}(6,6)=\left(\frac{148809c_3^{\,2}r_3}{124999559}-\frac{67030c_2c_3}{14688313}-\frac{r_2c_2^{\,2}}{840}+\frac{41869c_3^{\,2}}{12412927}+\frac{39404c_2^{\,2}}{11682127}+\frac{r_3c_2c_3r_2}{1\times10^{11}}+\right. \quad ;$$

$$\left.\frac{30493c_2c_3r_2}{25614121}-\frac{149557r_3c_2c_3}{125627881}+\frac{7154\left(r_3^{\,2}c_3^{\,2}+r_2^{\,2}c_2^{\,2}\right)}{24037439}\right)A$$

$$\mathbf{H}(6,10)=\left(\frac{15664r_2c_2}{13157761}+\frac{10054r_3^{\,2}c_3}{16890721}-\frac{r_3c_3}{280}+\frac{2976190c_2}{416666599}+\frac{r_1c_2}{1\times10^{10}}-\frac{r_1c_3}{2.5\times10^{10}}-\right. \quad ;$$

$$\left.\frac{35333c_3}{3124181}-\frac{189566r_3c_2}{159235441}-\frac{r_1r_3c_3}{1\times10^{11}}-\frac{r_1c_2r_2}{166666666666}+\frac{r_3c_2r_2}{2\times10^{11}}\right)A$$

$$\mathbf{H}(6,11)=\left(\frac{24001b_1r_3c_3}{40321679}-\frac{34523b_3c_3}{15817811}-\frac{\left(b_1c_2r_2+b_1r_1c_3\right)}{111111111111}-\frac{18884b_3r_3c_2}{31725119}+\right.$$

$$\frac{b_3r_3c_3}{333333333333}+\frac{7154r_3^{\,2}c_3b_3}{24037439}+\frac{5079\left(b_3c_2+b_1c_3\right)}{4654211}+\frac{\left(b_3c_2r_2+b_1r_1c_2\right)}{1\times10^{11}}- \quad ;$$

$$\left.\frac{3765b_1c_2}{3450109}-\frac{\left(b_1r_1c_2r_2+b_3r_3c_2r_2+b_1r_1c_3r_3\right)}{5\times10^{11}}\right)A$$

$$\mathbf{H}(6,12)=\left(\frac{7154r_3^{\,2}c_3^{\,2}}{24037439}-\frac{r_3c_2c_3r_2}{6.25\times10^{11}}+\frac{5079c_2c_3}{4654211}+\frac{5332c_1c_3}{4886051}-\frac{5778c_1c_2}{5294749}-\frac{34523c_3^{\,2}}{15817811}-\right.$$

$$\frac{r_3c_3^{\,2}}{1\times10^{12}}-\frac{21137r_3c_2c_3}{35510159}+\frac{25745c_1r_3c_3}{43251599}+\frac{c_1r_1c_2}{71428571428}+\frac{\left(c_2c_3r_2-c_1r_1c_3-c_1c_2r_2\right)}{1\times10^{11}}- \quad ;$$

$$\left.\frac{\left(c_1r_1c_2r_2+c_1r_1r_3c_3\right)}{5\times10^{11}}\right)A$$

$$\mathbf{H}(6,16)=\left(\frac{15664r_3c_3}{13157761}-\frac{r_3c_3r_2}{1\times10^{11}}-\frac{r_1c_2}{14285714285}-\frac{1879699c_3}{263157859}-\frac{114468r_2c_2}{32051041}+\right. \quad ;$$

$$\left.\frac{35333c_2}{3124181}-\frac{28077c_3r_2}{23584681}-\frac{10054r_2^{\,2}c_2}{16890721}+\frac{r_1c_2r_2}{111111111111}\right)A$$

$$\mathbf{H}(6,17)=\left(\frac{5079b_1c_2}{4654211}-\frac{24001b_1c_2r_2}{40321679}+\frac{c_2b_2r_2}{344827586206}+\frac{5332c_3b_2}{4886051}-\frac{34523c_2b_2}{15817811}-\right.$$

$$\frac{3765b_1c_3}{3450109}+\frac{7154r_2^{\,2}c_2b_2}{24037439}+\frac{24001c_3b_2r_2}{40321679}+\frac{b_1r_3c_3}{1.25\times10^{11}}-\frac{r_3c_3b_2}{58823529411}+\frac{b_1r_1c_2}{6.25\times10^{11}};$$

$$\left.-\frac{\left(b_1r_1r_3c_3+b_1r_1c_3\right)}{1\times10^{11}}-\frac{\left(b_1r_1c_2r_2+r_3c_3b_2r_2\right)}{5\times10^{11}}\right)A$$

$$\mathbf{H}(6,18)=\left(\frac{5618c_2c_3}{5148131}+\frac{9600c_1c_2}{8797091}-\frac{25745c_1c_2r_2}{43251599}-\frac{23347c_2^{\ 2}}{10697171}+\frac{15565c_2c_3r_2}{26149199}-\frac{4370c_1c_3}{4004509}+\right.$$

$$\left.\frac{7154r_2^{\ 2}c_2^{\ 2}}{24037439}-\frac{c_1r_1r_3c_3}{6.25\times10^{10}}-\frac{r_3c_2c_3}{76923076923}-\frac{\left(r_3c_2c_3r_2+c_1r_1c_2r_2\right)}{5\times10^{11}}+\frac{\left(c_1r_3c_3+c_1r_1c_2\text{-}c_1r_1c_3\right)}{1\times10^{11}}\right)A$$

$$\mathbf{H}(7,7)=\frac{A}{6};\ \mathbf{H}(7,9)=\frac{\left(b_1\text{-}b_3\right)A}{60};\ \mathbf{H}(7,13)=\frac{A}{12};\ \mathbf{H}(7,15)=\frac{\left(0.5b_2\text{-}b_1\right)A}{60};$$

$$\mathbf{H}(8,8)=\frac{A}{6};\ \mathbf{H}(8,9)=\frac{\left(c_1\text{-}c_3\right)A}{60};\ \mathbf{H}(8,14)=\frac{A}{12};\ \mathbf{H}(8,15)=\frac{\left(0.5c_2\text{-}c_1\right)A}{60};$$

$$\mathbf{H}(9,9)=\frac{\left(b_1^{\ 2}\text{-}b_1b_3+b_3^{\ 2}+c_1^{\ 2}\text{-}c_1c_3+c_3^{\ 2}\right)A}{360};\ \mathbf{H}(9,13)=\frac{\left(b_1\text{-}0.5b_3\right)A}{60};$$

$$\mathbf{H}(9,14)=\frac{\left(c_1\text{-}0.5c_3\right)A}{60};\ \mathbf{H}(9,15)=\frac{\left(b_1b_2\text{-}2b_1^{\ 2}\text{-}b_2b_3+b_1b_3+c_1c_2\text{-}2c_1^{\ 2}\text{-}c_3c_2+c_1c_3\right)A}{720};$$

$$\mathbf{H}(10,10)=\left(\frac{29}{140}\text{-}\frac{3809523r_3}{199999958}+\frac{4822181r}{253164503_1}+\frac{69213\left(r_3^{\ 2}+r_1^{\ 2}\right)}{58138921}\right)A;$$

$$\mathbf{H}(10,11)=\left(\frac{135281b_1r_3}{113636041}-\frac{1583080r_1b_1}{265957441}-\frac{676406r_3b_3}{113636209}+\frac{397500b_3}{17123077}+\frac{28077r_1b_3}{23584681}-\right.$$

$$\left.\frac{287299b_1}{12375957}+\frac{\left(b_1r_1r_3-r_1b_3r_3\right)}{142857142857}+\frac{10054\left(r_3^{\ 2}b_3-r_1^{\ 2}b_1\right)}{16890721}\right)A;$$

$$\mathbf{H}(10,12)=\left(\frac{362712c_3}{15624517}-\frac{287299c_1}{12375957}+\frac{148809r_1c_3}{124999559}-\frac{676406r_3c_3}{113636209}+\frac{28077c_1r_3}{23584681}-\right.$$

$$\left.\frac{2010939r_1c_1}{337837751}+\frac{c_1r_1r_3}{142857142857}-\frac{r_1r_3c_3}{111111111111}+\frac{10054\left(r_3^{\ 2}c_3-r_1^{\ 2}c_1\right)}{16890721}\right)A;$$

$$\mathbf{H}(10,16)=\left(\frac{788660}{12499517}\text{-}\frac{r_1}{2.5\times10^{10}}+\frac{r_3r_2}{1\times10^{11}}\text{-}\frac{148809r_1^{\ 2}}{124999559}\text{-}\frac{r_1r_2}{142857142857}\text{-}\right.$$

$$\left.\frac{r_1r_3}{333333333333}+\frac{297619\left(r_3\text{-}r_2\right)}{124999979}\right)A;$$

$$\mathbf{H}(10,17)=\left(\frac{14105r_2b_2}{11848201}\text{-}\frac{28077r_1b_2}{23584681}\text{-}\frac{1190476b_2}{166666641}\text{-}\frac{10054r_1^{\ 2}b_1}{16890721}+\frac{83650b_1}{7396421}\text{-}\right.$$

$$\left.\frac{156641r_1b_1}{43859481}+\frac{r_3b_2}{1\times10^{12}}+\frac{\left(r_3b_2r_2\text{-}r_1b_2r_2\right)}{1\times10^{11}}\right)A;$$

$$\mathbf{H}(10,18)=\left(\frac{74397c_1}{6578261}-\frac{46502r_1c_2}{39061681}-\frac{10054r_1^{2}c_1}{16890721}-\frac{1190476c_2}{166666641}-\frac{248015r_1c_1}{69444201}+\frac{15664r_2c_2}{13157761}+\frac{r_3c_2}{14285714285}-\frac{c_1r_3}{77519379844}+\frac{\left(c_1r_3r_1-r_1c_2r_2+r_3c_2r_2\right)}{1\times10^{11}}\right)A\;;$$

$$\mathbf{H}(11,11)=\left(\frac{b_1^{2}r_1}{840}-\frac{27017b_1b_3}{5920247}-\frac{561545b_3^{2}r_3}{471697799}-\frac{17610b_1r_1b_3}{14792401}+\frac{218837b_1b_3r_3}{183823081}+\frac{b_1r_1b_3r_3}{1\times10^{11}}+\frac{7154\left(r_3^{2}b_3^{2}+r_1^{2}b_1^{2}\right)}{24037439}+\frac{39404\left(b_1^{2}+b_3^{2}\right)}{11682127}\right)A\;;$$

$$\mathbf{H}(11,12)=\left(\frac{41869b_3c_3}{12412927}-\frac{930059b_3c_3r_3}{781249559}-\frac{13114c_1b_3}{5747353}-\frac{8008b_1c_3}{3509593}+\frac{39404c_1b_1}{11682127}+\frac{402187c_1b_1r_1}{337837079}+\frac{7154r_3^{2}b_3c_3}{24037439}-\frac{c_1b_3r_1r_3}{83333333333}-\frac{b_1r_1c_3r_3}{5\times10^{11}}+\frac{18884c_1b_3r_3}{31725119}+\frac{7154r_1^{2}b_1c_1}{24037439}-\frac{21137b_1r_1c_3}{35510159}+\frac{24001\left(b_1c_3r_3-c_1b_3r_1\right)}{40321679}\right)A\;;$$

$$\mathbf{H}(11,16)=\left(\frac{15664r_3b_3}{13157761}+\frac{3720r_1^{2}b_1}{6249601}+\frac{b_3}{140}-\frac{248015r_1b_1}{69444201}-\frac{b_1r_2}{1\times10^{10}}-\frac{95506b_1}{8444741}-\frac{b_3r_2}{1\times10^{12}}-\frac{105165r_1b_3}{88338601}+\frac{\left(r_1b_3r_3-b_1r_1r_2\right)}{1\times10^{11}}\right)A\;;$$

$$\mathbf{H}(11,17)=\left(\frac{9600b_1b_2}{8797091}-\frac{34523b_1^{2}}{15817811}-\frac{5778b_3b_2}{5294749}+\frac{5332b_1b_3}{4886051}+\frac{r_1b_1^{2}}{1\times10^{12}}-\frac{7154r_1^{2}b_1^{2}}{24037439}-\frac{24001b_1r_1b_3}{40321679}+\frac{b_1b_3r_3}{71428571428}+\frac{14308b_1r_1b_2}{24037439}+\frac{b_1r_1b_3r_3}{217391304347}-\frac{\left(b_1r_1b_2r_2+b_3r_3b_2r_2\right)}{5\times10^{11}}+\frac{\left(b_3b_2r_2-b_1b_2r_2-b_3r_3b_2\right)}{1\times10^{11}}\right)A\;;$$

$$\mathbf{H}(11,18)=\left(\frac{5079b_1c_2}{4654211}+\frac{b_3c_2r_2}{111111111111}+\frac{c_1b_3r_3}{71428571428}-\frac{15769c_1b_1}{7225069}+\frac{7154r_1^{2}b_1c_1}{24037439}+\frac{3462c_1b_3}{3172451}-\frac{5778b_3c_2}{5294749}-\frac{24001c_1r_1b_3}{40321679}-\frac{c_1b_1r_1}{142857142857}-\frac{b_1c_2r_2}{33333333333}-\frac{b_3r_3c_2}{6.25\times10^{10}}+\frac{24001b_1r_1c_2}{40321679}-\frac{1\left(b_3r_3c_2r_2+c_1r_1b_3r_3\right)}{5\times10^{11}}\right)A\;;$$

$$\mathbf{H}(12,12)=\left(\frac{7782r_1^{2}c_1^{2}}{26147519}+\frac{7154r_3^{2}c_3^{2}}{24037439}-\frac{r_3c_3^{2}}{840}+\frac{960061r_1c_1^{2}}{806451241}-\frac{11837c_1c_3}{2593847}+\frac{11763c_1r_3c_3}{9880921}+\frac{39404c_1^{2}}{11682127}+\frac{37211c_3^{2}}{11031967}-\frac{29235c_1c_3r_1}{24557401}+\frac{c_1r_3c_3r_1}{1\times10^{11}}\right)A\;;$$

$$\mathbf{H}(12,16)=\left(\frac{10054r_1^2c_1}{16890721}-\frac{74397c_1}{6578261}-\frac{1116071r_1c_1}{312499881}+\frac{13565r_3c_3}{11394601}+\frac{1190476c_3}{166666641}-\right.$$

$$\left.\frac{c_1r_2}{10638297872}+\frac{1c_3r_2}{24390243902}-\frac{189566r_1c_3}{159235441}-\frac{\left(c_1r_1r_2+r_3c_3r_2\right)}{1\times10^{11}}\right)A\;;$$

$$\mathbf{H}(12,17)=\left(\frac{5079b_1c_3}{4654211}-\frac{3765c_3b_2}{3450109}+\frac{8148c_1b_2}{7466531}+\frac{7154r_1^2c_1b_1}{24037439}-\frac{34523c_1b_1}{15817811}-\frac{r_3c_3b_2r_2}{2\times10^{11}}-\right.$$

$$\left.\frac{c_1b_2r_2}{5\times10^{10}}+\frac{c_3b_2r_2}{73529411764}+\frac{24001\left(c_1r_1b_2-b_1r_1c_3\right)}{40321679}+\frac{\left(b_1r_3c_3-r_3c_3b_2\right)}{1\times10^{11}}-\frac{\left(b_1r_1r_3c_3+c_1r_1b_2r_2\right)}{5\times10^{11}}\right)A$$

$$\mathbf{H}(12,18)=\left(\frac{7782r_1^2c_1^2}{26147519}+\frac{c_1r_3c_3}{1\times10^{11}}-\frac{r_3c_2c_3}{66666666666}+\frac{3462c_1c_3}{3172451}+\frac{5079c_1c_2}{4654211}-\frac{5778c_2c_3}{5294749}-\right.$$

$$\frac{8233c_1^2}{3772211}-\frac{1c_1c_2r_2}{76923076923}+\frac{1c_2c_3r_2}{71428571428}-\frac{24001c_1r_1c_3}{40321679}+\frac{21137c_1r_1c_2}{35510159}\;;$$

$$\left.-\frac{1\left(r_3c_2c_3r_2+c_1r_1r_3c_3+c_1r_1c_2r_2\right)}{5\times10^{11}}\right)A$$

$$\mathbf{H}(13,13)=\frac{A}{6}\;;\;\mathbf{H}(13,15)=\frac{\left(b_2-b_1\right)A}{60}\;;$$

$$\mathbf{H}(14,14)=\frac{A}{6}\;;\;\mathbf{H}(14,15)=\frac{\left(c_2-c_1\right)A}{60}\;;$$

$$\mathbf{H}(15,15)=\frac{\left(b_2^2-b_1b_2+b_1^2+c_2^2-c_1c_2+c_1^2\right)A}{360}\;;$$

$$\mathbf{H}(16,16)=\left(\frac{29}{140}+\frac{8105369r_2}{425531872}+\frac{69213r_2^2}{58138921}-\frac{4822181r_1}{253164503}+\frac{148809r_1^2}{124999559}\right)A\;;$$

$$\mathbf{H}(16,17)=\left(\frac{148809r_1b_2}{124999559}-\frac{74404r_2^2b_2}{124998719}+\frac{10054r_1^2b_1}{16890721}-\frac{268683b_2}{11574037}-\frac{567975r_1b_1}{95419801}-\right.$$

$$\left.\frac{676406r_2b_2}{113636209}+\frac{397500b_1}{17123077}+\frac{258799b_1r_2}{217391161}+\frac{\left(r_1b_2r_2-b_1r_1r_2\right)}{1\times10^{11}}\right)A\;;$$

$$\mathbf{H}(16,18)=\left(\frac{r_1c_2r_2}{1\times10^{11}}+\frac{28077c_1r_2}{23584681}-\frac{676406r_2c_2}{113636209}+\frac{69213r_1c_2}{58138921}-\frac{430085r_1c_1}{72254281}+\right.$$

$$\left.\frac{362712c_1}{15624517}-\frac{3720r_2^2c_2}{6249601}-\frac{308697c_2}{13297717}-\frac{c_1r_1r_2}{142857142857}+\frac{74404r_1^2c_1}{124998719}\right)A\;;$$

$$\mathbf{H}(17,17)=\left(\frac{95810b_1^2}{28404847}-\frac{b_1^2r_1}{840}-\frac{38540b_1b_2}{8445287}+\frac{402187r_2b_2^2}{337837079}+\frac{111824b_2^2}{33152527}+\right.$$

$$\left.\frac{28077\left(b_1r_1b_2-b_1b_2r_2\right)}{23584681}+\frac{7154\left(r_1^2b_1^2+r_2^2b_2^2\right)}{24037439}\right)A\;;$$

$$\mathbf{H}(17,18)=\left(\frac{11728c_1b_1}{3477007}-\frac{14910c_1b_2r_2}{25048799}+\frac{24001b_1r_1c_2}{40321679}-\frac{c_1b_1r_1}{840}-\frac{14308b_1c_2r_2}{24037439}-\right.$$

$$\frac{13114b_1c_2}{5747353}-\frac{12378}{5424793}c_1b_2+\frac{51542c_2b_2}{15280687}+\frac{148809c_2b_2r_2}{124999559}-\frac{b_1r_1c_2r_2}{5\times10^{11}}+ \qquad ;$$

$$\left.\frac{16281c_1r_1b_2}{27352079}-\frac{1c_1r_1b_2r_2}{6.25\times10^{11}}+\frac{7154\left(r_1^2b_1c_1+r_2^2b_2c_2\right)}{24037439}\right)A$$

$$\mathbf{H}(18,18)=\left(\frac{41869c_2^2}{12412927}-\frac{14252c_1c_2}{3123047}+\frac{39404c_1^2}{11682127}+\frac{c_1r_1c_2r_2}{166666666666}+\right.$$

$$\left.\frac{\left(r_2c_2^2-r_1c_1^2\right)}{840}+\frac{28077\left(c_1c_2r_1-c_1c_2r_2\right)}{23584681}+\frac{7154\left(r_1^2c_1^2+r_2^2c_2^2\right)}{24037439}\right)A \qquad .$$

APPENDIX B

Based in the bending element stiffness matrix evaluated by Yuqiu, et al. (1993), the membrane stiffness matrix for GPL-T9 element in explicit form is given by:

$$[K_m]=[R_m]^T[Q_m][R_m]$$

where

$$[R_m]=\frac{1}{4A}\begin{bmatrix} 2b_1 & 0 & 0 & 2b_2 & 0 & -b_2b_3 & 2b_3 & 0 & b_2b_3 \\ 2b_1 & 0 & b_1b_3 & 2b_2 & 0 & 0 & 2b_3 & 0 & -b_1b_3 \\ 2b_1 & 0 & -b_1b_2 & 2b_2 & 0 & b_1b_2 & 2b_3 & 0 & 0 \\ 0 & 2c_1 & 0 & 0 & 2c_2 & -c_2c_3 & 0 & 2c_3 & c_2c_3 \\ 0 & 2c_1 & c_1c_3 & 0 & 2c_2 & 0 & 0 & 2c_3 & -c_1c_3 \\ 0 & 2c_1 & -c_1c_2 & 0 & 2c_2 & c_1c_2 & 0 & 2c_3 & 0 \\ 2c_1 & 2b_1 & 0 & 2c_2 & 2b_2 & -(c_2b_3+b_2c_3) & 2c_3 & 2b_3 & (c_2b_3+b_2c_3) \\ 2c_1 & 2b_1 & (c_1b_3+b_1c_3) & 2c_2 & 2b_2 & 0 & 2c_3 & 2b_3 & -(c_1b_3+b_1c_3) \\ 2c_1 & 2b_1 & -(c_1b_2+b_1c_2) & 2c_2 & 2b_2 & (c_1b_2+b_1c_2) & 2c_3 & 2b_3 & 0 \end{bmatrix};$$

$$[Q_m]=\begin{bmatrix} D_{m11}[P] & D_{m12}[P] & D_{m16}[P] \\ D_{m12}[P] & D_{m22}[P] & D_{m26}[P] \\ D_{m16}[P] & D_{m26}[P] & D_{m66}[P] \end{bmatrix}; \qquad [P]=\frac{A}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

The membrane-bending coupling element stiffness matrix in the explicit form is defined as follows:

$$[K_{mb}]=[R_m]^T[Q_{mb}][R_b]$$

$$\left[Q_{mb}\right]=\begin{bmatrix} D_{mb_{11}}\left[P\right] & D_{mb_{12}}\left[P\right] & D_{mb_{16}}\left[P\right] \\ D_{mb_{12}}\left[P\right] & D_{mb_{22}}\left[P\right] & D_{mb_{26}}\left[P\right] \\ D_{mb_{16}}\left[P\right] & D_{mb_{26}}\left[P\right] & D_{mb_{66}}\left[P\right] \end{bmatrix}; \qquad \left[R_b\right]=\left[C_b\right]\left[A_b\right];$$

$$\left[C_b\right]=-\frac{1}{4A^2}\begin{bmatrix} 2b_1b_2 & 2b_2b_3 & 2b_3b_1 & 3b_1^2 & -3b_2^2 & -3b_3^2 \\ 2b_1b_2 & 2b_2b_3 & 2b_3b_1 & -3b_1^2 & 3b_2^2 & -3b_3^2 \\ 2b_1b_2 & 2b_2b_3 & 2b_3b_1 & -3b_1^2 & -3b_2^2 & 3b_3^2 \\ 2c_1c_2 & 2c_2c_3 & 2c_3c_1 & 3c_1^2 & -3c_2^2 & -3c_3^2 \\ 2c_1c_2 & 2c_2c_3 & 2c_3c_1 & -3c_1^2 & 3c_2^2 & -3c_3^2 \\ 2c_1c_2 & 2c_2c_3 & 2c_3c_1 & -3c_1^2 & -3c_2^2 & 3c_3^2 \\ 2\left(b_1c_2+b_2c_1\right) & 2\left(b_2c_3+b_3c_2\right) & 2\left(b_3c_1+b_1c_3\right) & 6b_1c_1 & -6b_2c_2 & -6b_3c_3 \\ 2\left(b_1c_2+b_2c_1\right) & 2\left(b_2c_3+b_3c_2\right) & 2\left(b_3c_1+b_1c_3\right) & -6b_1c_1 & 6b_2c_2 & -6b_3c_3 \\ 2\left(b_1c_2+b_2c_1\right) & 2\left(b_2c_3+b_3c_2\right) & 2\left(b_3c_1+b_1c_3\right) & -6b_1c_1 & -6b_2c_2 & 6b_3c_3 \end{bmatrix};$$

$$\left[A_b\right]=\begin{bmatrix} 0 & 0 & 0 & -2 & 1-r_2 & 1+r_3 \\ -\frac{1}{2}b_3 & 0 & \frac{1}{2}b_2 & -\frac{1}{2}\left(b_2-b_3\right) & -\frac{1}{2}\left(r_2b_2+b_3\right) & -\frac{1}{2}\left(r_3b_3-b_2\right) \\ -\frac{1}{2}c_3 & 0 & \frac{1}{2}c_2 & -\frac{1}{2}\left(c_2-c_3\right) & -\frac{1}{2}\left(r_2c_2+c_3\right) & -\frac{1}{2}\left(r_3c_3-c_2\right) \\ 0 & 0 & 0 & 1+r_1 & -2 & 1-r_3 \\ \frac{1}{2}b_3 & -\frac{1}{2}b_1 & 0 & -\frac{1}{2}\left(r_1b_1-b_3\right) & -\frac{1}{2}\left(b_3-b_1\right) & -\frac{1}{2}\left(r_3b_3+b_1\right) \\ \frac{1}{2}c_3 & -\frac{1}{2}c_1 & 0 & -\frac{1}{2}\left(r_1c_1-c_3\right) & -\frac{1}{2}\left(c_3-c_1\right) & -\frac{1}{2}\left(r_3c_3+c_1\right) \\ 0 & 0 & 0 & 1-r_1 & 1+r_2 & -2 \\ 0 & \frac{1}{2}b_1 & -\frac{1}{2}b_2 & -\frac{1}{2}\left(r_1b_1+b_2\right) & -\frac{1}{2}\left(r_2b_2-b_1\right) & -\frac{1}{2}\left(b_1-b_2\right) \\ 0 & \frac{1}{2}c_1 & -\frac{1}{2}c_2 & -\frac{1}{2}\left(r_1c_1+c_2\right) & -\frac{1}{2}\left(r_2c_2-c_1\right) & -\frac{1}{2}\left(c_1-c_2\right) \end{bmatrix}^T.$$