NUMERICAL SIMULATION AND CONSTRUCTAL THEORY APPLIED FOR GEOMETRIC OPTIMIZATION OF THIN PERFORATED PLATES SUBJECT TO ELASTIC BUCKLING

Anderson L. G. Correia

Daniel Helbig

Mauro de V. Real

Elizaldo D. dos Santos

Liércio A. Isoldi

anderson_luis_88@hotmail.com

daniel.helbig@gmail.com

mauroreal@furg.br

elizaldosantos@furg.br

liercioisoldi@furg.br

Universidade Federal do Rio Grande (FURG) - Escola de Engenharia (EE)

Itália Ave. km 8, 96203-900, Rio Grande do Sul, Rio Grande, Brazil

Luiz A. O. Rocha

luizrocha@mecanica.ufrgs.br

Universidade Federal do Rio Grande do Sul (UFRGS) - Departamento de Engenharia Mecânica (DEMEC)

Sarmento Leite St. nº 425, 90050-170, Rio Grande do Sul, Porto Alegre, Brazil

Abstract. *Many elements in engineering are formed by thin plates. Hulls and decks of ships are examples of application. These elements can have holes that serve as inspection port, access or even to weight reduction. The presence of holes causes a redistribution of the membrane stresses in the plate, significantly altering their stability. In this paper the Bejan's Constructal Theory was employed to discover the best geometry of thin perforated plates submitted to elastic buckling phenomenon. To study this behavior simply supported rectangular plates with a centered elliptical perforation were analyzed. The purpose was to obtain the optimal geometry which maximizes the critical buckling load. For this, the degrees of freedom H/L (ratio between width and length of the plate) and* H_0/L_0 *(ratio between the characteristic dimensions of the hole) were varied. Moreover, different values of hole volume fraction* ϕ *(ratio between the perforation volume and the massive plate volume) were also investigated. A computational modeling, based on the Finite Element Method (FEM), was used for assessing the plate buckling load. The results showed that Constructal Design can be* *employed not only in the heat transfer and fluid flow problems, but also to define the best shapes in solid mechanics problems.*

Keywords: *Numerical Simulation, Constructal Theory, Elastic buckling*

1 INTRODUCTION

It is possible to state that improving systems configuration for achieving better performance is the major goal in engineering. In the past, the scientific and technical knowledge combined with practice and intuition has guided engineers in the design of man-made systems for specific purposes. Soon after, the advent of the computational tools has permitted to simulate and evaluate flow architectures with many degrees of freedom. However, while system performance was analyzed and evaluated on a scientific basis, system design was kept at the level of art (Bejan & Lorente, 2006a).

The Constructal Theory was created by Adrian Bejan, in 1997, when a new geometric solution philosophy was applied to the conductive cooling of electronics (Bejan, 1997, 2000). These studies have a significant importance because they played a basic and starting point role for the extension and application of Constructal Theory to problems in engineering and other branches of science (Bejan & Lorente, 2008; Ghodoossi, 2004). Moreover, Constructal Theory has been employed to explain deterministically the generation of shapes in nature (Bejan, 2000).

The Constructal Theory states that: “for a flow system to persist in time (to survive) it must evolve in such a way that it provides easier and easier access to the currents that flow through it”. It is not only a principle from which geometric shape and structure are deduced, but also an engineering method for optimizing the paths for flows through finite-size open systems (Bejan & Lorente, 2006b).

This is a major step toward making system design a science. This theory indicates that if a system is free to morph under global constraints, the better flow architecture is the one that minimizes the global flow resistances, or maximizes the global flow access. A basic outcome of the Constructal Theory is that system shape and internal flow architecture do not develop by chance, but they result from the permanent struggle for better performance and therefore must evolve in time. As in engineered systems, in nature the competition is permanent (e.g., river basins, global circulations, trees and animals morph and improve in time under changing constraints) (Bejan & Lorente, 2006a).

Concerning the engineering problems, the applicability of Constructal Design (Constructal Theory for optimization of several systems, e.g., engineering) has been discussed largely in the recent literature: Azad & Amidpour (2011), Beyene & Peffley (2009), Kang et al. (2010) and Kim et al. (2010, 2011). As can be observed, these studies have been dominantly applied for the study of fluid mechanics and heat transfer.

However, few studies in the field of mechanics of materials employing the Constructal Design have been developed. In this subject, it is worthy to mention the studies of Lorente & Bejan (2002) and Lorente et al. (2010). The first paper draws attention to a specific class of thermal design problems, in which the system architecture is derived from a combination of heat transfer and mechanical strength considerations. In the latter work, it was studied the analogy between the geometric configuring of heat and fluid flow and the configuring of the stress distribution (“flow of stresses”). Recently, other studies using the Constructal Design method in pure solid mechanics applications were developed, see Rocha et al. (2012), Isoldi et al. (2013) and Rocha et al. (2013). In these works elastic buckling and stress concentration of thin perforated plates were analyzed.

In this context, the present work aims to apply the Constructal Design method to define the best shape of a thin perforated plate subjected to elastic buckling. To do so, a rectangular simply supported plate with a centered elliptical perforation was considered. The degree of freedom H/L (ratio between height and length of the plate) and the degree of freedom H_0/L_0 (ratio between the characteristic dimensions of the hole) are optimized for different hole volume fractions ϕ (ratio between the perforation volume and the massive plate volume). The objective is to maximize the critical buckling load.

A numerical approach was adopted for assessing the plate buckling load, and the Lanczos method was applied to the solution of the corresponding eigenvalue problem. The computational model, based on the Finite Element Method (FEM), was verified comparing its results with analytical solutions and with other numerical results.

2 BUCKLING OF PLATES

Buckling is an instability phenomenon that can occur if a slender and thin-walled plate (plane or curved) is subjected to axial compression. At a certain given critical load the plate will buckle very sudden in the out-of-plane transverse direction (Åkesson, 2007).

The problem of the elastic buckling of a simply supported rectangular plate of length L, width H, thickness t, and subjected to an axial load P, as can be seen in Fig. 1, has great importance in structural design.

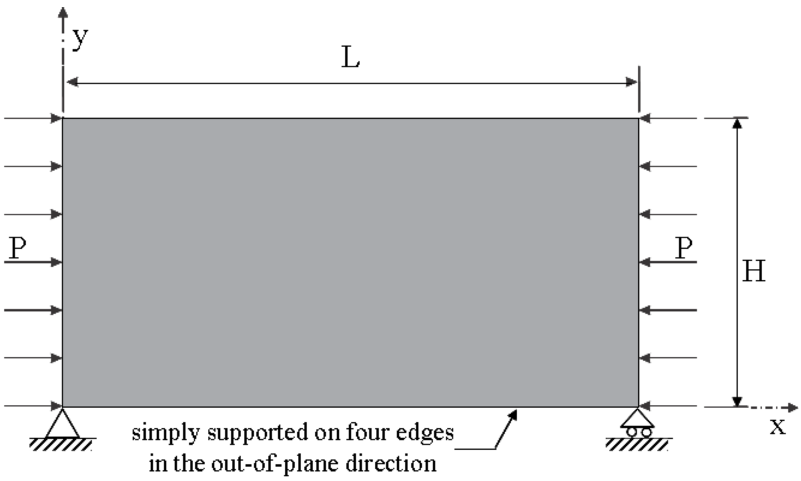

Figure 1. Rectangular plate subject to uniaxial compressive load

The general expression for the critical buckling stress is (Åkesson, 2007; El-Sawy & Nazmy, 2001; Wang et al., 2005):

$$\sigma_{cr} = k\frac{\pi^2 E}{12\left(1-\nu^2\right)\left(\frac{H}{t}\right)^2} \tag{1}$$

where π is the mathematical constant, E and ν are the Young's modulus and the Poisson's ratio of the plate material, respectively, the quotient H/t is the slenderness (ratio) of the plate and k is the buckling coefficient, given by:

$$k = \left(m\frac{H}{L} + \frac{1}{m}\frac{L}{H} \right)^2 \tag{2}$$

being m the number of half waves that occur in the plate's longitudinal direction at buckling, defining the buckling mode of the plate. The buckling coefficient varies depending on the type of stress distribution, and on the quotient between the length and the width of the plate (k has its lowest value for pure axial loading in compression, which also gives the lowest value for the critical buckling stress). An important characteristic of the buckling is that the instability may occur at a stress level that is substantially lower than the material yield stress, σ_y.

When the load P (see Fig. 1) reaches a certain critical value, expressed as P_{cr} (or σ_{cr} for the critical stress), the plate buckles and collapses. For any given axial loading below this critical value, it is possible to apply an additional transversal force without the occurrence of buckling. The closer the axial load is to the critical buckling load, the less the ability to carry an additional transversal horizontal loading becomes. At exactly the critical buckling load, this ability becomes zero – the plate is then barely able to just carry the axial load. The critical buckling load is defined by the product of critical buckling stress and thickness of the plate:

$$P_{cr} = k\frac{\pi^2 E t^3}{12H^2\left(1-\nu^2\right)} \tag{3}$$

3 COMPUTATIONAL MODEL

In the present study the critical buckling load of plates was determined using the general-purpose finite element program ANSYS®. The approach adopted for buckling analysis was the eigenvalue buckling (linear). This numerical procedure is used for calculating the theoretical buckling load of a linear elastic structure. Since it assumes the structure exhibits linearly elastic behavior, the predicted buckling loads are overestimated (Madenci & Guven, 2006).

Therefore, if the component is expected to exhibit structural instability, the search for the load that causes structural bifurcation is referred to as a buckling load analysis. Because the buckling load is not known a priori, the finite element equilibrium equations for this type of analysis involve the solution of homogeneous algebraic equations whose lowest eigenvalue corresponds to the buckling load, and the associated eigenvector represents the primary buckling mode (Madenci & Guven, 2006).

The strain formulation used in the analysis includes both the linear and nonlinear terms. Thus, the total stiffness matrix, $[K]$, is obtained by summing the conventional stiffness matrix for small deformation, $[K_E]$, with another matrix, $[K_G]$, which is the so-called geometrical stiffness matrix. The matrix $[K_G]$ depends not only on the geometry but also on the initial internal forces (stresses) existing at the start of the loading step, $\{P_0\}$. Therefore the total stiffness matrix of the plate with load level $\{P_0\}$ can be written as (Przeminieck, 1985):

$$[K]=[K_E]+[K_G] \tag{4}$$

When the load reaches the level of $\{P\} = \lambda\{P_0\}$, where λ is a scalar, the stiffness matrix can be defined as:

$$[K]=[K_E]+\lambda[K_G] \tag{5}$$

Now, the governing equilibrium equations for the plate behavior can be written as:

$$\left[[K_E]+\lambda[K_G]\right]\{U\}=\lambda\{P_0\} \tag{6}$$

where $\{U\}$ is the total displacement vector, that may therefore be determined from:

$$\{U\}=\left[[K_E]+\lambda[K_G]\right]^{-1}\lambda\{P_0\} \tag{7}$$

At buckling, the plate exhibits a large increase in its displacements with no increase in the load. From the mathematical definition of the matrix inverse as the adjoint matrix divided by the determinant of the coefficients it is possible to note that the displacements $\{U\}$ tend to infinity when:

$$\det\ \left[[K_E]+\lambda[K_G]\right]=0 \tag{8}$$

Equation (8) represents an eigenvalue problem, which when solved provides the lowest eigenvalue, λ_1, that corresponds to the critical load level $\{P_{cr}\} = \lambda_1\{P_0\}$ at which buckling occurs. In addition, the associated scaled displacement vector $\{U\}$ defines the mode shape at buckling. In the finite element program ANSYS, the eigenvalue problem is solved by using the Lanczos numerical method (ANSYS, 2005).

In all numerical simulations throughout this work the ANSYS® SHELL93 reduced integration eight-node thin shell element (Fig. 2) was adopted. This element has six degrees-of-freedom at each node: three translations (u,v,w) and three rotations $(\theta_x,\theta_y,\theta_z)$ (ANSYS, 2005).

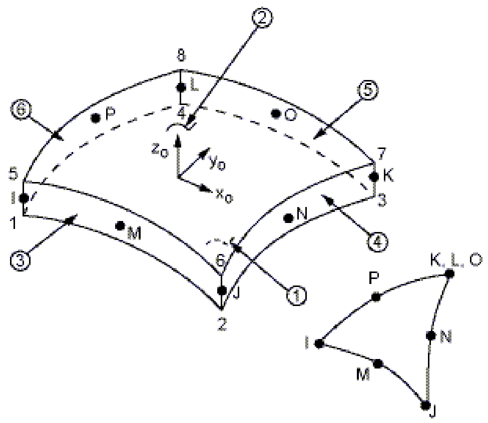

Figure 2. ANSYS SHELL93 8-node element geometry

3.1 Computational model verification

To verify the computational modeling the critical load of non perforated plates are carried out, and these numerical results are compared with the analytical solutions given by Eq. (3). Considering the Fig. 1, steel plates with material properties of E = 210 GPa, ν = 0.30, and σ_y = 250 MPa are simulated. The domain is dicretized by means of triangular elements with side size of 0.05 m. The dimensions, number of half waves, and buckling coefficient for each plate are presented in Table 1, as well as, the analytical and numerical results for the critical buckling load.

Table 1. Values and simulations of model verification process

Value	Plate 1	Plate 2	Plate 3
H (m)	1.43	1.00	0.77
L (m)	1.40	2.00	2.60
t (m)	0.01	0.01	0.01
m	1	2	3
k	4.00	4.00	4.06
Analytical Solution (kN/m)	372.16	759.20	1301.39
Numerical Solution (kN/m)	370.31	755.30	1294.00
Difference (%)	-0.50	-0.51	-0.57

Observing the difference between analytical and numerical results in Table 1 it is possible to affirm that the computational model to obtain the critical buckling load of a simply supported massive plate is verified.

Another model verification is performed taking into account thin perforated steel plates. The same plate 2 (see Table 1) used in the first verification is studied, however centered circular holes are considered. In Table 2 the results for the critical buckling load are compared with those obtained by the numerical study developed by El-Sawy & Nazmy (2001).

Table 2. Comparison of critical buckling load for plate with centered circular hole

Hole Diameter (m)	P_{cr} (kN/m) Reference	P_{cr} (kN/m) Present work	Difference (%)
0.10	766.19	763.56	-0.34
0.20	789.36	786.50	-0.36
0.30	825.08	820.87	-0.51
0.40	849.26	847.78	-0.17
0.50	901.54	898.79	-0.31
0.60	986.46	981.22	-0.53

Again an excellent agreement is obtained, being -0.53% the maximal difference encountered, i.e., the computational model proposed is verified.

4 CONSTRUCTAL DESIGN

Most of the activity in the field of constructal theory and design is devoted to the development of tree-shaped architectures for fluid flow and heat transfer. However, it is possible to consider the solid structures as flow systems that are configured and morph so that they facilitate the flow of stresses. To look at stresses as flow is quite unusual but it is effective when the objective is to discover the best configuration of the stressed volume (Lorente et al., 2010).

Based on this, the geometric elastic buckling optimization for thin plates with centered elliptical hole was investigated. Keeping constant the plate thickness (t), the characteristic dimensions of the perforation (H_0 and L_0, as can be seen in Fig. 3) can vary, as well as, the plate external dimensions (H and L). So, the degrees of freedom are defined as: H_0/L_0 and H/L for constant areas of the hole. In addition, the hole variation is governed by the parameter called hole volume fraction (ϕ). This parameter represents the relation between the hole volume (V_0) and the total plate volume without hole (V). Therefore, for the plate with a centered elliptical perforation (Fig. 3) the hole volume fraction is defined as:

$$\phi = \frac{V_0}{V} = \frac{(\pi H_0 L_0 t)/4}{HLt} = \frac{\pi H_0 L_0}{4HL} \tag{9}$$

where π is the mathematical constant; H_0 and L_0 the hole characteristic dimensions of hole in y and x directions, respectively; H is the height of plate, L is the length and t is the plate thickness.

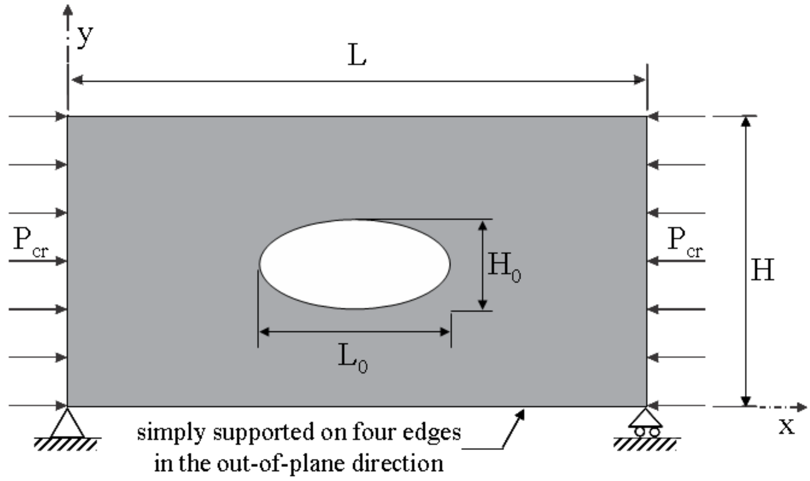

Figure 3. Plate with centered elliptical hole

The objective is to determine the optimal plate geometry that is characterized by the maximization of critical buckling load for the perforated plate. For this, based on Constructal Design, the variables of the problem were considered dimensionless:

$$\tilde{x}, \tilde{y}, \tilde{t}, \tilde{H}, \tilde{L}, \tilde{H}_0, \tilde{L}_0 = \frac{x, y, t, H, L, H_0, L_0}{A^{1/2}} \quad (10)$$

being A the area of plate without hole defined as:

$$A = HL \quad (11)$$

In summary, six different values are considered for the hole volume fraction parameter (ϕ). For each ϕ three different ratios H/L were considered and hence for each H/L several values of H_0/L_0 were investigated, as illustrated in Fig. 4.

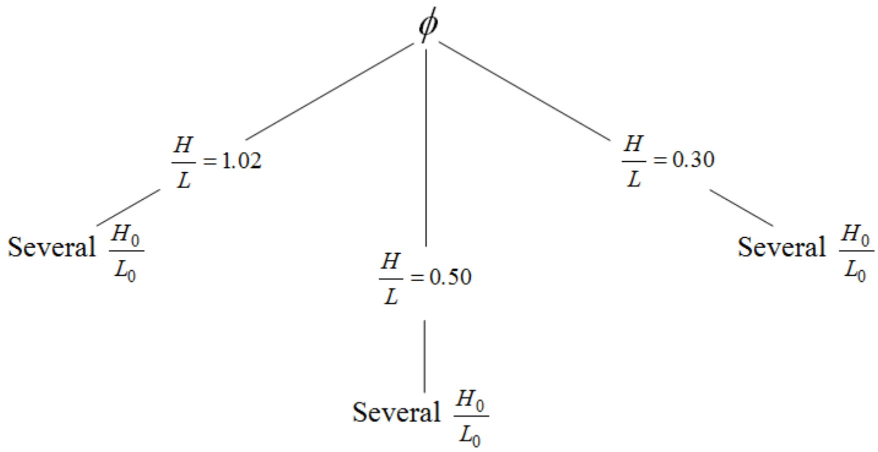

Figure 4. Flow chart illustrating the optimization process

5 RESULTS AND DISCUSSION

To define the best shape of a rectangular simply supported perforated thin plate subjected to elastic buckling, by means the Constructal Design method, six values are adopted for the hole volume fraction: ϕ = 0.02, 0.05, 0.08, 0.15, 0.20, and 0.25. For each ϕ three steel plates with different ratio H/L are considered (see Fig. 4). The external dimensions H and L are the same presented in Table 1. Besides, several ratios H_0/L_0 are investigated for each plate.

It is worth to emphasize that the highest numerical critical buckling load presented in Table 1 is used as reference to obtain the dimensionless value for the critical bucking load of perforated plates.

In a first optimization level it is possible to find optimal geometries related to the H_0/L_0 variation. So, for each H/L and for each ϕ a best shape for the perforated plate can be obtained. For instance, Fig. 5 shows the variation of critical buckling load regarding the ratio H_0/L_0, for plates with H/L = 0.50 and for the six value of hole volume fraction mentioned above.

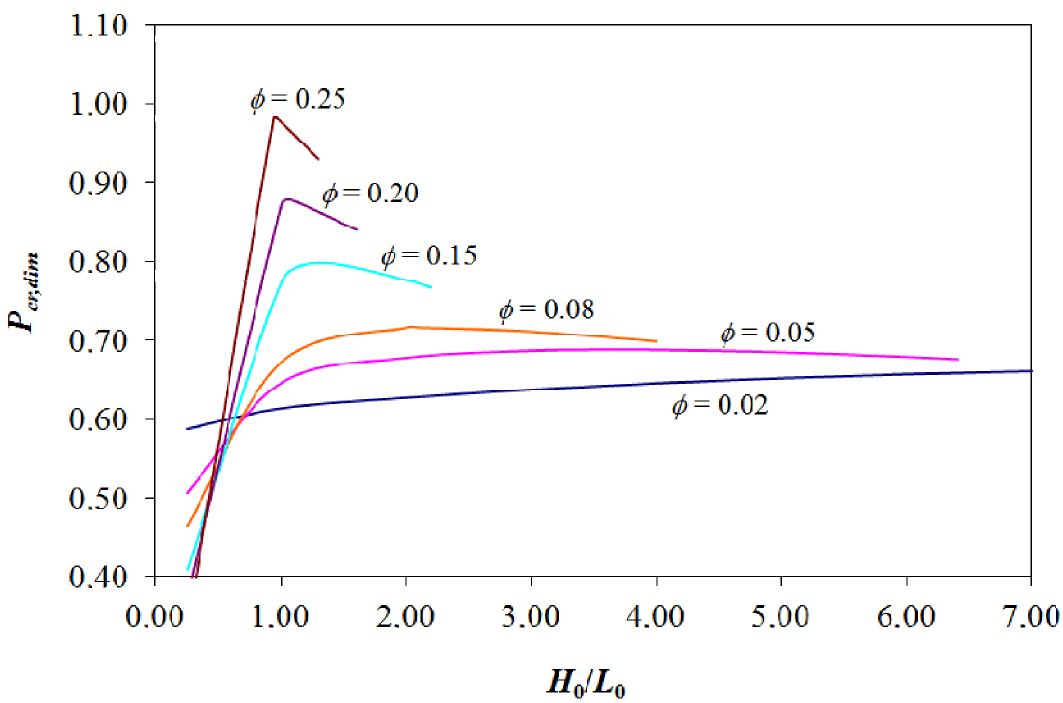

Figure 5. Effect of H_0/L_0 over $P_{cr,dim}$ for various values of ϕ and fixed ratio of H/L = 0.50

In Fig. 5 it is possible to observe that for the highest values of ϕ there are intermediate well defined optimal geometries, i.e., a geometry that conducts to the maximum critical buckling load. As the value of ϕ decreases, the optimal value of $(H_0/L_0)_o$ increases until the lowest value of ϕ = 0.02 where the highest value of H_0/L_0 leads to the best mechanical performance.

Figure 6 also presents the values of $P_{cr,dim}$ versus the H_0/L_0 variation, however in this graph all ratios H/L for a specific hole volume fraction (ϕ = 0.20) are exhibited.

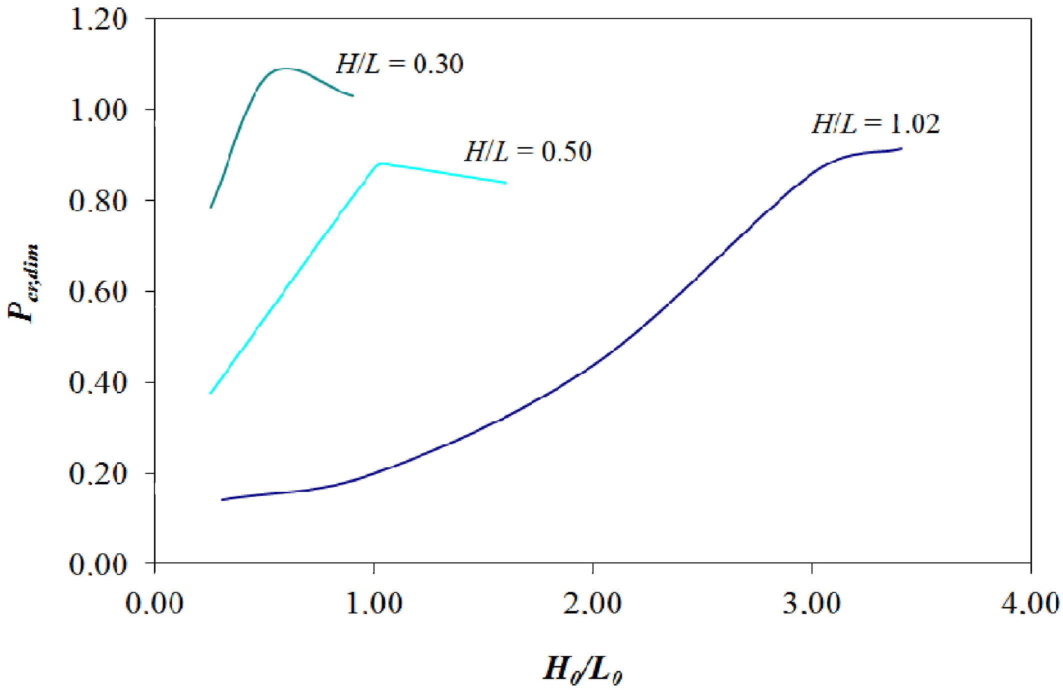

Figure 6. Effect of H_0/L_0 over $P_{cr,dim}$ for various values of H/L and fixed value of ϕ = 0.20

Figure 6 indicates that comparatively the plate that presents the higher critical buckling has the ratio H/L = 0.30. However, the best shape for the plate with H/L = 1.02 reaches a superior value of maximum critical buckling if compared with that obtained for the plate H/L = 0.50. In addition, the worst case for the plate H/L = 0.30 has a value for the critical load lower than those obtained for several cases of plates H/L = 0.50 and H/L = 1.02, showing the importance of the shape perforation in the performance of the plates subjecting to buckling.

After that a second optimization level can be defined starting from the best shapes encountered in the first optimization level. Now, for each hole volume fraction proposed the maximum value of critical load for each ratio H/L is achieved, as can be seen in Fig. 7.

One can note in Fig. 7 that for values of ϕ largest than 0.08 the critical buckling load for the plate H/L = 1.02 achieves a superior level than the critical load of plate H/L = 0.50. However this trend is not observed for values of ϕ smaller than 0.08. Moreover, it is possible to identify that the dimensionless maximum critical buckling loads for all cases have a value around 1.05, except the plate H/L = 0.30 which achieves 1.15.

The best geometry that conducts to the maximum critical buckling load obtained in Fig. 7 are depicted in Fig. 8. In this figure it is possible to observe the effect of H/L over the once optimized ratio of $(H_0/L_0)_o$ for all studied hole volume fractions (ϕ). In general, it is observed an augmentation of the optimal ratio of $(H_0/L_0)_o$ with the increase of the ratio (H/L), with exception for ϕ = 0.02 where the ratio H/L is almost insensitive over the optimal ratio of $(H_0/L_0)_o$.

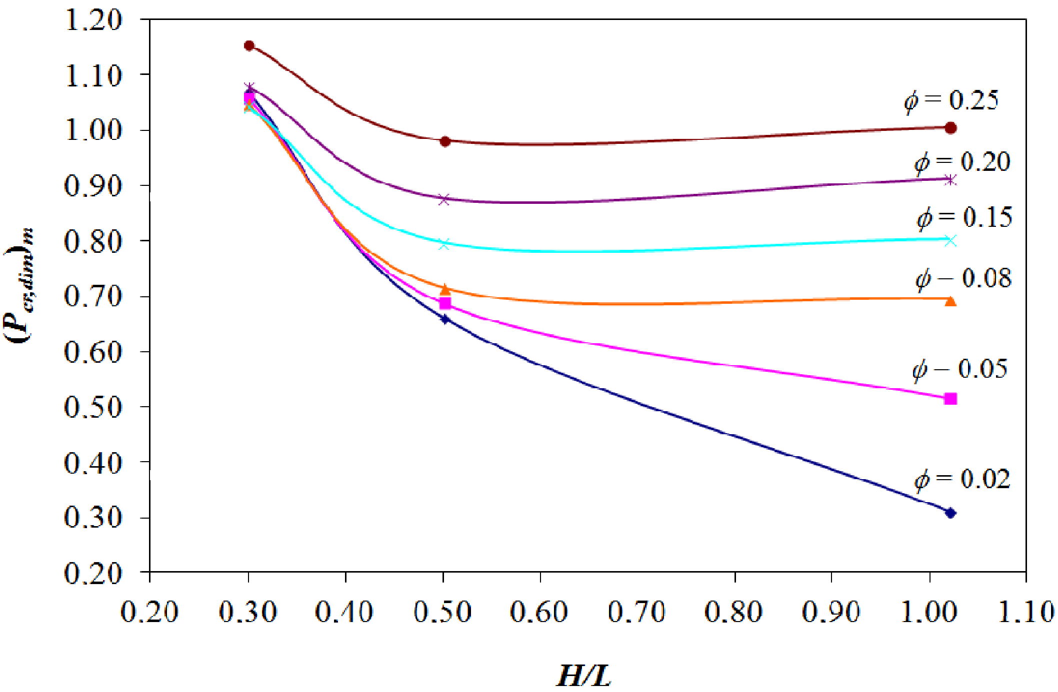

Figure 7. The effect of *H*/*L* over the once maximized ($P_{cr,dim}$)$_m$

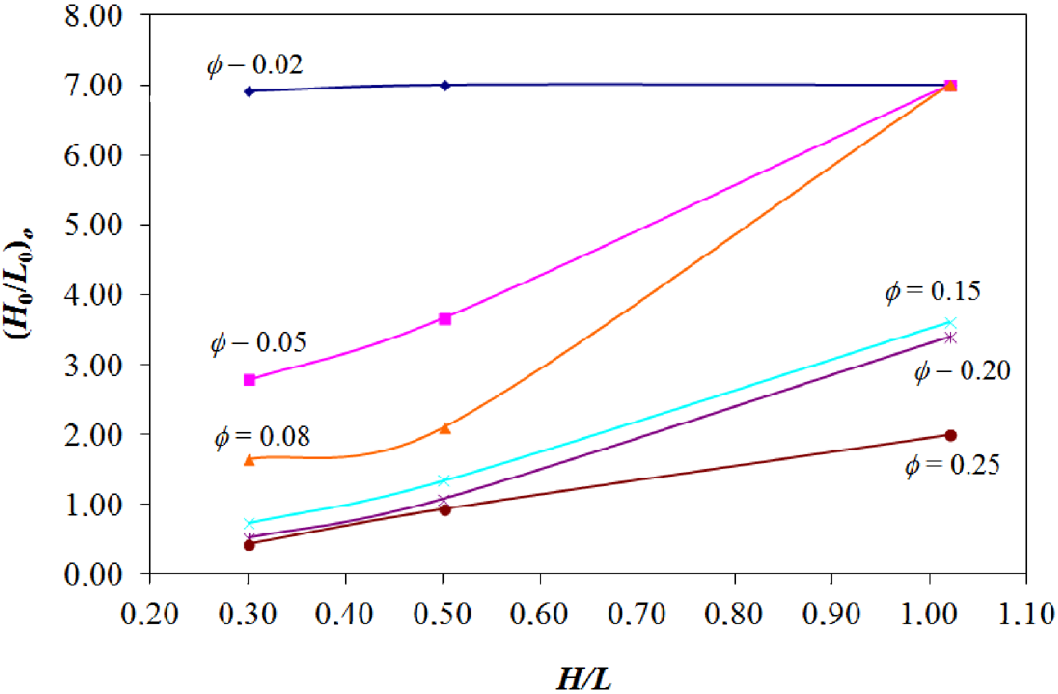

Figure 8. The effect of H/L over the once optimized ratio of (H_0/L_0)$_o$

Finally, the influence of the hole volume fraction ϕ in the geometrical optimization process was determined by grouping the best results of Fig. 7 and its correspondent ratio H_0/L_0 (showed in Fig. 8), generating the Fig. 9 and Fig. 10, respectively. In other words, it is obtained the twice maximized dimensionless critical load $(P_{cr,dim})_{mm}$ and the twice optimized ratio of $(H_0/L_0)_{oo}$ as function of the perforation volume fraction (ϕ).

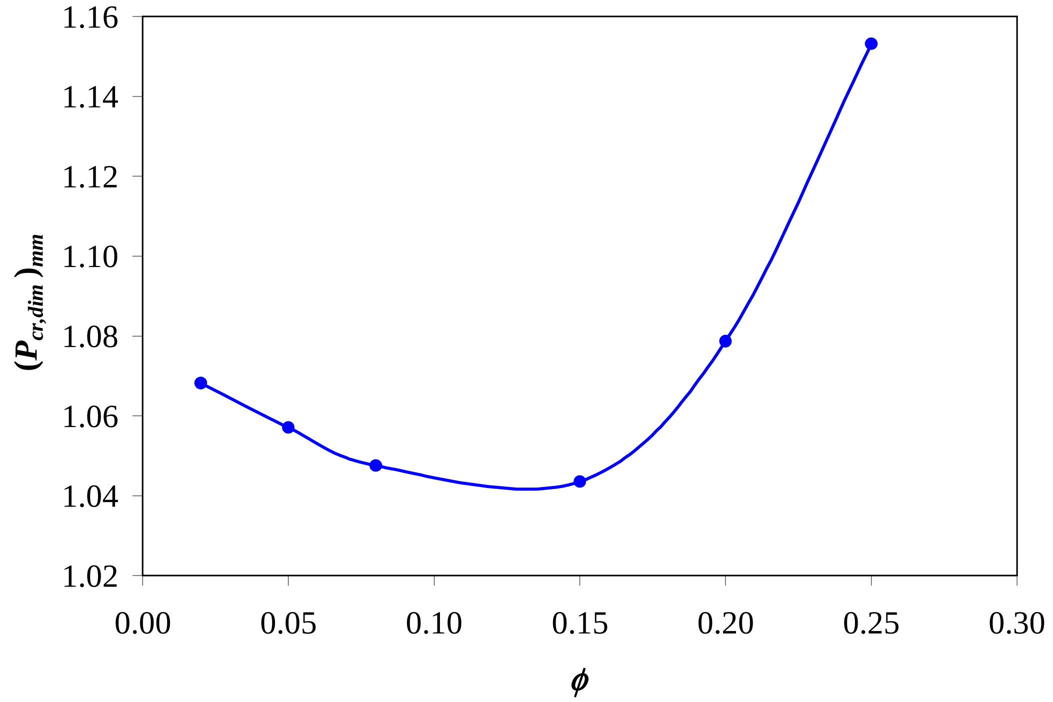

Figure 9. Effect of ϕ over the twice maximized $(P_{cr,dim})_{mm}$

Figure 9 indicates that there is a hole volume fraction (ϕ = 0.15) that conducts to a minimum value among the maximum critical buckling loads. Other indication of Fig. 9 is that the maximum critical buckling load among all cases was obtained for a ϕ = 0.25.

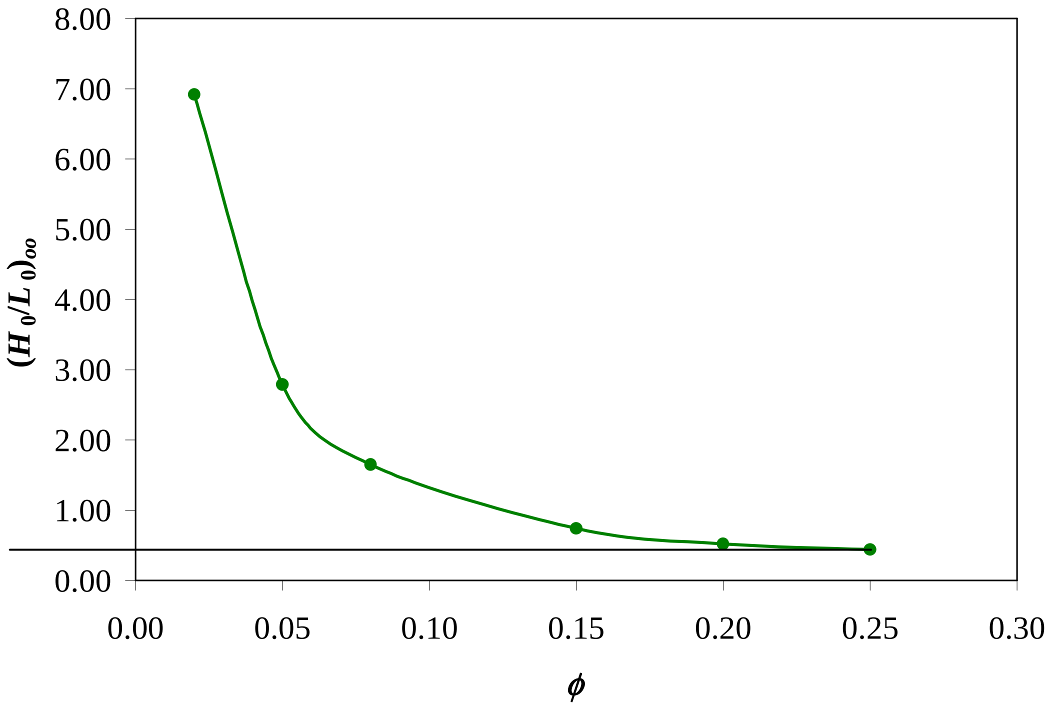

Figure 10. Effect of ϕ over the twice maximized $(H_0/L_0)_{oo}$

In Fig. 10 one can observe how the hole volume fraction is related with the H_0/L_0, existing a ϕ value augmentation accompanied by a decrease of the ratio H_0/L_0. It is also noticed that, for the highest values of ϕ, the twice optimized ratio of $(H_0/L_0)_{oo}$ tends to a constant value of $(H_0/L_0)_{oo} = 0.4$.

6 CONCLUSIONS

The importance of a better understanding about the mechanical behavior of thin steel plate structural components is evident due its wide application in main branches of engineering, mainly if these plates have a perforation that causes meaningful changes in its mechanical behavior. Moreover, the search for structural elements with optimized geometries is a constant requirement in engineering applications.

In this context, the main goal of the present work was to develop a numerical study of thin perforated steel plates subjected to elastic buckling, aiming do obtain its best shapes, i.e., the shapes that can support the higher values for the critical buckling load. To do so a computational model based on the Finite Element Method (FEM) was employed to solve the several geometries for the perforated plate which were defined according to Constructal Design methodology.

The results indicated that the variation of the shape and dimensions of the plate and hole can define geometries with excellent performances as well as geometries which lead to an undesirable mechanical behavior. The best shape nearly 5 times better than the worst shape. Moreover, the optimization results showed that there is no universal shape that leads to the best mechanical performance, being required the employment of Constructal Design method to seek for the best shapes.

In addition, once again, it was possible to show that the Constructal Design can be employed in mechanic of materials with the same efficiency that it has been broadly used in fluid mechanics and heat transfer applications.

ACKNOWLEDGEMENTS

The authors thank FURG, CAPES and CNPq by the support.

REFERENCES

ANSYS., 2005. *User's manual (version 10.0)*. Swanson Analysis System Inc, Houston.

Åkesson, B., 2007. *Plate buckling in bridges and other structures.* Taylor & Francis.

Azad, A. V., & Amidpour, M., 2011. Economic optimization of shell and tube heat exchanger based on constructal theory. *Energy,* vol. 36, pp. 1087-1096.

Bejan, A., 1997. Constructal-theory network of conducting paths for cooling a heat generating volume. *International Journal of Heat and Mass Transfer*, vol. 40, n. 4, pp. 799–816.

Bejan, A., 2000. *Shape and structure, from engineering to nature.* Cambridge University Press, Cambridge.

Bejan, A., & Lorente, S., 2006a. The Constructal Law (La Loi Constructale). *International Journal of Heat and Mass Transfer,* vol. 49, pp. 445-445.

Bejan, A., & Lorente, S., 2006b. Constructal Theory of generation of configuration in nature and engineering. *Journal of Applied Physics*, vol. 100, pp. 041301.

Bejan, A., & Lorente, S., 2008. *Design with Constructal Theory*. Wiley, Hoboken.

Beyene, A., & Peffley, J., 2009. Constructal Theory, adaptive motion, and their theoretical application to low-speed turbine design. *Journal of Energy Engineering-ASCE*, vol. 135, n. 4, pp. 112-118.

El-Sawy, K. M., & Nazmy, A. S., 2001. Effect of aspect ratio on the elastic buckling of uniaxially loaded plates with eccentric holes. *Thin-Walled Structures*, vol. 39 , pp. 983–998.

Ghodoossi, L., 2004. Conceptual study on constructal theory. *Energy Conversion and Management*, vol. 45, pp. 1379–1395.

Isoldi, L. A., Real, M. V., Correia, A. L. G., Vaz, J., dos Santos, E. D., & Rocha, L. A. O., 2013. Flow of stress: Constructal Design of perforated plates subjected to tension or buckling (Chapter 12). In Rocha, L. A. O., Lorente, S., & Bejan, A., eds, *Constructal Law and the Unifying Principle of Design*, pp. 195-217. Springer.

Kang, D. -H., Lorente, S., & Bejan, A., 2010. Constructal dendritic configuration for the radiation heating of a solid stream. *Journal of Applied Physics*, vol. 107, pp. 114910.

Kim, Y., Lorente, S., & Bejan, A., 2010. Constructal multi-tube configuration for natural and forced convection in cross-flow. *International Journal of Heat and Mass Transfer*, vol. 53, pp. 5121-5128.

Kim, Y., Lorente, S., & Bejan, A., 2011. Steam generator structure: continuous model and constructal design. *International Journal of Energy Research*, vol. 35, pp. 336-345.

Lorente, S., & Bejan, A., 2002. Combined 'flow and strength' geometric optimization: internal structure in a vertical insulating wall with air cavities and prescribed strength. *International Journal of Heat and Mass Transfer*, vol. 45, pp. 3313-3320.

Lorente, S., Lee, J., & Bejan, A., 2010. The "flow of stresses" concept: the analogy between mechanical strength and heat convection. *International Journal of Heat and Mass Transfer*, vol. 53, pp. 2963-2968.

Madenci, E., & Guven, I., 2006. *The Finite Element Method and Applications in Engineering Using ANSYS®*. Springer.

Przemieniecki, J. S., 1985. *Theory of Matrix Structural Analysis.* Ed. Dover Publications.

Rocha, L. A. O., Real, M. V., Correia, A. L. G., Vaz, J., dos Santos, E. D., & Isoldi L. A., 2012. Geometric optimization based on the constructal design of perforated thin plates subject to buckling. *Computational Thermal Sciences*, vol. 4, n. 2, pp. 119-129.

Rocha, L. A. O., Isoldi L. A., Real, M. V., dos Santos, E. D., Correia, A. L. G., Lorenzini, G., & Biserni, C., 2013. Constructal Design Applied to the Elastic Buckling of Thin Plates with Holes, *Central European Journal of Engineering*, vol. 3, pp. 475–483.

Wang, C. M., Wang, C. Y., & Reddy, J. N., 2005. *Exact solutions for buckling of structural members.* CRC Press.