Analysis of Finite Volumes and Integral Transform Solutions for Thermally Developing Non-Newtonian Fluid Flow

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Abstract

The current work provides a comparison between two different methodologies for solving convection-diffusion problems: the Generalized Integral Transform Technique (GITT) and the Finite Volumes Method (FVM). The problem of thermally developing laminar flow of non-Newtonian fluids between parallel plates is selected for illustrating purposes. Both solutions focus on the transformation of a partial-differential formulation into an ordinary-differential form, either through integral transformation or discretization of the directional variable transversal to the flow. The resulting ODE systems are solved analytically and comparison results are presented, indicating advantages and disadvantages of each methodology. Once comparisons are performed advantages and disadvantages of each methodology are discussed. The results indicate that, in general, the integral transform technique presents a better convergence rate.

Key-words: Integral Transform, Finite Volumes, Non-Newtonian Fluid, Parallel Plates

1 Introduction

Despite many fluids present Newtonian behavior, non-Newtonian fluids, whose apparent viscosity varies with shear rate, are of great importance to several industrial applications. The models of Generalized Newtonian fluids, specially Power-Law fluid and Bingham plastic, due to its simplicity and ability to represent various problems of engineering interest is used in this work.

Simulation of heat transfer in fluid flow usually requires a computational approach for solving the associated partial differential equations. One such technique, which has been growing in popularity, is the so called Generalized Integral Transform Technique (GITT) (Cotta, 1993), a method which uses hybrid schemes, part analytical and part numerical. Another popular method is the Finite Volume Method (FVM) (Patankar, 1980; Maliska, 1995), which appears as widely used option to a variety of convection-diffusion problems, due to its conservative nature and ease of application. Nevertheless, as with any discrete method, approximations to integrals derivatives in terms of nodal points on a computational domain are necessary, resulting in a solution error, which decays with grid refinement.

The Integral Transform approach deals with expansions of the sought solution in terms of a basis of infinite orthogonal eigenfunctions, maintaining the solution process always within a continuum domain. However, since the infinite series representation must be truncated for any possible solution implementation, a truncation error is involved. Then again, this error decreases as the number of terms are increased and the solution converges to a final value. Due to the nature of the series representation error estimates can be easily obtained from this method, which results in a better control of the global solution error. The usual drawback associated with this approach is the elaborate mathematical manipulation; however this effort can be considerably minimized with the employment of symbolical computation (Wolfram, 2003).

Interesting applications of the GITT include a variety of convection-diffusion problems. For heat transfer in internal forced convection different investigations were carried out employing the GITT. Among the recent advancements for these type of problems, one should mention (Macêdo et al., 2000; Nacimento et al., 2002, 2006), which deals with non-Newtonian flows in circular shaped ducts, (Maia et al., 2006), which presents a solution for non-Newtonian flows in elliptical cross-section ducts, and (Lima et al., 2007), which investigates the MHD flow and heat transfer within parallel-plates channels. For flow in ducts of arbitrary geometry, some particular solutions have been developed (Aparecido and Cotta, 1990; Ding and Manglik, 1996; Barbuto and Cotta, 1997; Guerrero et al., 2000); nonetheless, a general methodology was described in (Sphaier and Cotta, 2000, 2002), being potentially promising for these types of geometries.

Both discrete and spectral approaches have been demonstrated to serve as effective methodologies for solving convection-diffusion problems, but there is a relative lack of comparative studies. This paper provides a performance comparison between the Generalized Integral Transform Technique and the Finite Volumes Method to the problem of thermally developing laminar duct flow. Simulation results using both methodologies are presented, as well as a preliminary study of the combined application of the spectral and discrete approaches.

A recent investigation compared the performance of GITT and FVM solutions for steady thermally developing laminar channel flow (Chalhub et al., 2008). however, only results of the simplified cases with large Péclet values were examined. Numerical results are calculated using the Mathematica system.

In order to perform a consistent comparison between the two methodologies, the FVM solution is carried out by discretizing in a single spatial variable. With this strategy a coupled ODE system for the other variable is obtained. This system is solved using analytical and numerical methods, as done for the GITT solution. This approach is the so-called Method of Lines, which in the classical sense analytical solutions for the transformed system are obtained. A generalization to this method can be made if the solution to the resulting ODE system is handled numerically, and the approach becomes the Numerical Method of Lines (Schiesser, 1991). An advantage of this approach is that the ODE system is usually solved by routines that enable user prescribed error control. A recent investigation (Sphaier and Worek, 2008) employed this approach for periodic heat and mass exchangers, which was showed to be very effective.

2 Mathematical Formulation

2.1 Problem presentation

The studied problem is that of heat transfer in steady incompressible laminar flow between two parallel plates. The flow is considered hydrodynamically developed, but thermally developing. The problem is given by the following dimensionless equations:

$$\frac{1}{2}\,u^*\,\frac{\partial\theta}{\partial\xi} \,=\, \mathrm{Pe}_H^{-2}\frac{\partial^2\theta}{\partial\xi^2} \,+\, \frac{\partial^2\theta}{\partial\eta^2}, \tag{1}$$

$$\theta(\xi,1) \,=\, 0, \qquad \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} = \,0, \tag{2}$$

$$\theta(0,\eta) \,=\, 1, \qquad \left(\frac{\partial\theta}{\partial\xi}\right)_{\xi\to\infty} = \,0, \tag{3}$$

where the dimensionless quantities are given by:

$$\theta = \frac{T-T_0}{T_{in}-T_0}, \qquad \eta \,=\, \frac{y}{H/2}, \qquad \xi \,=\, \frac{x}{L}, \tag{4}$$

and the value of L is chosen from a scale analysis of the thermal entry length:

$$L \,=\, \frac{H}{2}\,\mathrm{Pe}_H, \qquad \text{with} \qquad \mathrm{Pe}_H \,=\, \frac{\bar{u}\,H}{\alpha}. \tag{5}$$

Two different non-Newtonian fluids are considered, leading to the following fully developed velocity profiles.

Power-law:

$$u^* \,=\, \frac{u}{\bar{u}} \,=\, \frac{1+2n}{1+n}\,(1-\eta^{1+1/n}),\; 0\leq\eta\leq 1 \tag{6}$$

Bingham plastic:

$$u^* \,=\, \frac{u}{\bar{u}} \,=\, \frac{3}{2}\left[\frac{(1-\eta^2)-2\,\eta_0(1-\eta)}{1-\frac{3}{2}\,\eta_0+\frac{1}{2}\,\eta_0^3}\right],\; \eta_0\leq\eta\leq 1 \tag{7}$$

$$u^* \,=\, \frac{u}{\bar{u}} \,=\, \frac{3}{2+\eta_0},\; 0\leq\eta<\eta_0 \tag{8}$$

where n is the power-law exponent and η_0 is a Bingham fluid parameter related to the yield stress.

The Nusselt number in terms of the dimensionless variables is given by:

$$\mathrm{Nu}_{D_H} = \frac{-4\,(\partial\theta/\partial\eta)_{\eta=1}}{\int_0^1 u^*\theta \mathrm{d}\eta}. \tag{9}$$

For large Péclet values the axial diffusion term can be neglected and the energy equation is reduced to:

$$\frac{1}{2}\, u^* \, \frac{\partial\theta}{\partial\xi} = \frac{\partial^2\theta}{\partial\eta^2}, \tag{10}$$

and only the boundary condition at the inlet ($\xi = 0$) is necessary for the axial direction.

2.2 Finite Volumes Method

The solution of the studied problem via finite volumes is accomplished by integrating eq. (1) within a finite volume of height $\Delta\eta = 1/J$ and employing second-order approximations for integration and interpolation, which leads to the following discretized system:

$$-\mathrm{Pe}_H^{-2}\,\frac{\mathrm{d}^2\hat{\theta}_j}{\mathrm{d}\xi^2} + \frac{1}{2}\,\hat{u}_j^*\,\frac{\mathrm{d}\hat{\theta}_j}{\mathrm{d}\xi} = F_j(\xi), \tag{11}$$

$$\hat{\theta}_j(0) = 1, \qquad \left(\frac{\mathrm{d}\hat{\theta}_j}{\mathrm{d}\xi}\right)_{\xi=\xi_{\max}} = 0, \tag{12}$$

for $j = 1, 2, \ldots, J$. The F-functions, which carry all the η-discretization information, are given by:

$$F_j(\xi) = \frac{\hat{\theta}_{j+1} - \hat{\theta}_j}{\Delta\eta^2}, \qquad \text{for} \qquad j = 1, \tag{13}$$

$$F_j(\xi) = \frac{\hat{\theta}_{j+1} - 2\,\hat{\theta}_j + \hat{\theta}_{j-1}}{\Delta\eta^2}, \qquad \text{for} \qquad 1 < j < J, \tag{14}$$

$$F_j(\xi) = \frac{\hat{\theta}_{j-1} - 3\,\hat{\theta}_j}{\Delta\eta^2}, \qquad \text{for} \qquad j = J, \tag{15}$$

For cases with small to moderate Péclet numbers, this system is solved numerically using the **NDSolve** function available in the *Mathematica* software. Using the obtained solutions, the Nusselt number is then calculated from eq. (9), by computing the derivative and integral numerically.

2.2.1 Solutions for large Péclet

For cases of large Péclet numbers, the previous system is simplified, by the elimination of the second derivative terms. Considering these cases and denoting the discrete variables with an over-hat, the discretized equations for all volumes are written in compact form:

$$\frac{\mathrm{d}\hat{\theta}_j}{\mathrm{d}\xi} = \frac{2}{\hat{u}_j^*} F_j(\xi), \qquad \text{for} \qquad j = 1, 2, \ldots, J \tag{16}$$

this initial-value system is solved numerically using the *Mathematica* function **NDSolve**.

Equation (16) can also be written in matrix form, as displayed below:

$$\hat{\boldsymbol{\theta}}'(\xi) = \hat{\boldsymbol{M}}\, \hat{\boldsymbol{\theta}}(\xi) \tag{17}$$

where the coefficients of $\hat{\boldsymbol{M}}$ are given by:

- for $j = 1$:

$$\hat{M}_{j,j} = -\frac{2}{\hat{u}_j^* \, \Delta\eta^2}, \qquad \hat{M}_{j,j+1} = \frac{2}{\hat{u}_j^* \, \Delta\eta^2}, \tag{18}$$

- for $1 < j < J$:

$$\hat{M}_{j,j-1} = \frac{2}{\hat{u}_j^* \, \Delta\eta^2}, \qquad \hat{M}_{j,j} = -\frac{4}{\hat{u}_j^* \, \Delta\eta^2}, \tag{19}$$

$$\hat{M}_{j,j+1} = \frac{2}{\hat{u}_j^* \, \Delta\eta^2}, \tag{20}$$

- for $j = J$:

$$\hat{M}_{j,j-1} = \frac{2}{\hat{u}_j^* \, \Delta\eta^2}, \qquad \hat{M}_{j,j} = -\frac{6}{\hat{u}_j^* \, \Delta\eta^2}, \tag{21}$$

and the remaining $\hat{M}_{j,k}$ coefficients are zero. The analytical solution for the dimensionless temperature at the discrete points can be then written in the following form:

$$\hat{\boldsymbol{\theta}}(\xi) = \hat{\boldsymbol{C}}(\xi) \cdot \hat{\boldsymbol{b}}, \qquad \text{with} \qquad \hat{\boldsymbol{C}}(\xi) = \exp(\hat{\boldsymbol{M}}\, \xi). \tag{22}$$

where $\hat{\boldsymbol{C}}$ is a matrix exponential Greenberg (1998). The coefficients of $\hat{\boldsymbol{b}}$ are the discrete values of the inlet conditions. Hence, $\hat{b}_j = \hat{\theta}_j(0) = 1$.

2.3 Generalized Integral Transform Technique

The integral transform solution of the considered problem is accomplished employing the Generalized Integral Transform Technique Cotta (1993). The solution of the problem is started by defining the transformation pair:

$$\text{Transform} \quad \Longrightarrow \quad \bar{\theta}_n(\xi) = \int_0^1 \theta(\xi,\, \eta) Y_n(\eta)\, \mathrm{d}\eta, \tag{23}$$

$$\text{Inversion} \quad \Longrightarrow \quad \theta(\xi,\, \eta) = \sum_{n=1}^{\infty} \frac{\bar{\theta}_n(\xi)\, Y_n(\eta)}{N(\lambda_n)}, \tag{24}$$

where Y_n's are orthogonal solutions to a Sturm-Liouville eigenvalue problem. For the convection-diffusion problem considered in this work, the following eigenvalue problem is selected:

$$Y_n''(\eta) \,+\, \lambda_n^2 Y_n(\eta) = 0, \qquad \text{for} \qquad 0 \leq \eta \leq 1, \tag{25}$$

$$Y'(0) = 0, \qquad\qquad Y(1) \,=\, 0. \tag{26}$$

The previous problem leads to infinite nontrivial solutions in the form:

$$Y_n(\eta) \,=\, \cos(\lambda_n\, \eta), \qquad \lambda_n \,=\, \left(n - \frac{1}{2}\right)\pi, \tag{27}$$

for $n = 1, 2, 3, \ldots$. The norm of the Y_n eigenfunctions are given by:

$$N(\lambda_n) \,=\, \int_0^1 Y_n^2(\eta)\, \mathrm{d}\eta \,=\, \frac{1}{2}. \tag{28}$$

The transformation of the given problem is accomplished by multiplying eq. (1) by Y_n, integrating within $0 \leq \eta \leq 1$, and applying the inversion formula (24) to the non-transformable terms. This process yields:

$$\mathrm{Pe}_H^{-2}\, \bar{\theta}_n''(\xi) \,-\, \frac{1}{2} \sum_{m=1}^{\infty} A_{n,m}\, \bar{\theta}_m'(\xi) \,-\, \lambda_n^2 \bar{\theta}_n(\xi) \,=\, 0, \tag{29}$$

with the following boundary conditions:

$$\bar{\theta}_n(0) \,=\, b_n \,=\, \int_0^1 Y_n(\eta)\,\mathrm{d}\eta, \qquad \lim_{\xi\to\infty} \bar{\theta}'_n(\xi) \,=\, 0, \tag{30}$$

where the $A_{n,m}$ coefficients are given by:

$$A_{n,m} = \frac{1}{N(\lambda_m)} \int_0^1 u^*(\eta)\, Y_n(\eta)\, Y_m(\eta)\, \mathrm{d}\eta \tag{31}$$

For a general case of small to moderate Péclet numbers, this boundary value problem is solved numerically using the *Mathematica* function **NDSolve** and the dimensionless temperature is calculated using the inversion formula (24). For simpler cases, as described below, fully analytical solutions can be obtained. Regardless of the simplification considered, the Nusselt number is computed from the following expression:

$$\mathrm{Nu}_{D_H} = \frac{-4 \sum\limits_{n=1}^{\infty} \bar{\theta}_n / N(\lambda_n)\, Y'_n(1)}{\sum\limits_{n=1}^{\infty} \bar{\theta}_n / N(\lambda_n)\, \int_0^1 u^* Y_n \mathrm{d}\eta} \tag{32}$$

2.3.1 Solution for large Péclet

A simplified first-order form of system (29, 30), is obtained for large Péclet numbers:

$$\frac{1}{2} \sum_{m=1}^{\infty} A_{n,m}\, \bar{\theta}'_m(\xi) \,+\, \lambda_n^2 \bar{\theta}_n(\xi) \,=\, 0. \tag{33}$$

where only the first boundary condition (30) is required. This system can be solved numerically using the **NDSolve** routine. Nevertheless, an alternative analytical solution can be obtained. Writing the simplified system in matrix form yields

$$\boldsymbol{A}\, \bar{\boldsymbol{\theta}}'(\xi) \,+\, \boldsymbol{D}^*\, \bar{\boldsymbol{\theta}}(\xi) \,=\, \boldsymbol{0}, \qquad \bar{\boldsymbol{\theta}}(0) \,=\, \boldsymbol{b}, \tag{34}$$

where $\boldsymbol{A}$ is given by the integral coefficients in equation (31) and $\boldsymbol{D}^*$ is given by

$$D^*_{n,n} \,=\, 2\, \lambda_n^2\, \delta_{n,m}, \tag{35}$$

leads to the following closed-form analytical solution:

$$\bar{\boldsymbol{\theta}}(\xi) = \boldsymbol{C}\,\boldsymbol{b}, \qquad \text{with} \qquad \boldsymbol{C} = \exp\left(-\boldsymbol{A}^{-1}\boldsymbol{D}^{*}\,\xi\right) \tag{36}$$

and the temperature profile is obtained using the inversion formula (24).

3 Results and Discussion

Before calculating the results using the non-Newtonian velocity profiles, the proposed algorithms were validated using the traditional Hagen-Poiseulle profile obtained for laminar Newtonian flow. The results were in perfect agreement with literature data.

Table 1 presents the results for different power-law fluids calculated by the Finite Volumes Method. Numerical integration with **NDSolve** was employed for solving the resulting ODE system. As can be seen, in general, the convergence rate is spatially uniform and does not depend on the n number.

Table 2 presents similar Nusselt results calculated for different Bingham fluids. As one can observe, the convergence rate is becomes worse as η_0 is increase for positions in the neighborhood of the channel entrance; nevertheless, the convergence becomes better for larger η_0 away from the channel entrance.

Table 3 displays the Nusselt results calculated with the Integral Transform Technique. As can be seen, the convergence is worse for positions near the entrance, where more than 100 terms are needed for obtaining six significant figures. Nonetheless, a much better convergence rate is obtained for other positions. The same observation can be made to the table 4, where the Nusselt is calculated for a Bingham plastic.

Tables 5 and 6 present a comparison of the estitimated relative error for selected cases. The columns indicate relative error estimates at each position, calculated, for the Finite Volumes Method, by

$$\epsilon = \frac{|\mathrm{Nu}_{2\,I} - \mathrm{Nu}_{I}|}{\mathrm{Nu}_{2I}}, \tag{37}$$

and for Generalized Integral Transform Technique by:

$$\epsilon = \frac{|\mathrm{Nu}_{i_{\max}+10} - \mathrm{Nu}_{i_{\max}}|}{\mathrm{Nu}_{i_{\max}+10}}. \tag{38}$$

Table 1: Nusselt number for FVM with power-law fluids.

	I	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
$n = 0.5$	25	26.6409	12.8973	8.04642	7.93629
	50	26.8171	12.9068	8.04848	7.93899
	100	26.8413	12.908	8.04891	7.93959
	200	26.8444	12.9082	8.049	7.93972
	400	26.8448	12.9082	8.04902	7.93975
	800	26.8448	12.9082	8.04903	7.93976
	1600	26.8448	12.9082	8.04903	7.93976
$n = 1$	25	24.5393	12.0064	7.62977	7.53754
	50	24.6685	12.0134	7.63164	7.53998
	100	24.6857	12.0144	7.63204	7.54054
	200	24.6879	12.0145	7.63212	7.54066
	400	24.6882	12.0145	7.63214	7.54069
	800	24.6882	12.0145	7.63215	7.54070
	1600	24.6882	12.0145	7.63215	7.54070
$n = 2$	25	23.2731	11.4551	7.35664	7.27489
	50	23.3782	11.4608	7.35841	7.27721
	100	23.3921	11.4616	7.35879	7.27773
	200	23.3938	11.4617	7.35887	7.27786
	400	23.394	11.4618	7.3589	7.27789
	800	23.3941	11.4618	7.3589	7.2779
	1600	23.3941	11.4618	7.3589	7.2779
$n = 10$	25	22.1113	10.9413	7.09375	7.02135
	50	22.1972	10.9461	7.09539	7.0235
	100	22.2084	10.9468	7.09575	7.02399
	200	22.2099	10.9469	7.09584	7.02411
	400	22.2101	10.9469	7.09586	7.02414
	800	22.2101	10.9469	7.09586	7.02415
	1600	22.2101	10.9469	7.09586	7.02415
$n = 50$	25	21.8591	10.8289	7.03536	6.96496
	50	21.9412	10.8335	7.03696	6.96706
	100	21.9519	10.8342	7.03731	6.96754
	200	21.9533	10.8343	7.03739	6.96765
	400	21.9535	10.8343	7.03741	6.96768
	800	21.9535	10.8343	7.03742	6.96769
	1600	21.9535	10.8343	7.03742	6.96769

These quantities provide an estimate of the local error.

Table 2: Nusselt number for FVM with Bingham fluids.

	I	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
$\eta_0 = 0$	25	24.5393	12.0064	7.62977	7.53753
	50	24.6685	12.0134	7.63164	7.53998
	100	24.6857	12.0144	7.63204	7.54054
	200	24.6879	12.0145	7.63212	7.54066
	400	24.6882	12.0145	7.63214	7.54069
	800	24.6882	12.0145	7.63215	7.5407
	1600	24.6882	12.0145	7.63215	7.5407
	3200	24.6882	12.0145	7.63215	7.5407
$\eta_0 = 0.25$	25	25.8499	12.6181	7.96895	7.86526
	50	26.0076	12.6267	7.97097	7.86788
	100	26.0289	12.6279	7.97138	7.86846
	200	26.0316	12.628	7.97147	7.86859
	400	26.0319	12.628	7.97149	7.86862
	800	26.0319	12.628	7.9715	7.86863
	1600	26.0319	12.628	7.9715	7.86863
	3200	26.0319	12.628	7.9715	7.86863
$\eta_0 = 0.5$	25	28.3295	13.7627	8.51352	8.3803
	50	28.5519	13.7749	8.51583	8.3833
	100	28.5829	13.7765	8.51628	8.38395
	200	28.5868	13.7767	8.51638	8.38409
	400	28.5873	13.7767	8.5164	8.38412
	800	28.5873	13.7767	8.5164	8.38413
	1600	28.5874	13.7767	8.5164	8.38414
	3200	28.5874	13.7767	8.5164	8.38414
$\eta_0 = 0.75$	25	33.7999	15.9977	9.23878	9.06493
	50	34.2116	16.0212	9.242	9.06893
	100	34.2757	16.0243	9.24257	9.06973
	200	34.2839	16.0247	9.24267	9.0699
	400	34.2849	16.0247	9.2427	9.06994
	800	34.285	16.0247	9.2427	9.06994
	1600	34.285	16.0247	9.2427	9.06995
	3200	34.285	16.0247	9.2427	9.06995

Table 3: Nusselt number for GITT with power-law fluids.

	$n_{\max}$	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
$n = 0.5$	10	28.0266	12.9309	8.05013	7.94062
	20	26.9060	12.9111	8.04917	7.93987
	30	26.8630	12.9091	8.04907	7.93980
	40	26.8525	12.9086	8.04904	7.93978
	50	26.8488	12.9084	8.04904	7.93977
	70	26.8462	12.9083	8.04903	7.93977
	80	26.8458	12.9082	8.04903	7.93977
	90	26.8455	12.9082	8.04903	7.93976
	100	26.8453	12.9082	8.04903	7.93976
$n = 1$	10	25.2425	12.0299	7.63289	7.54128
	20	24.7301	12.0165	7.63224	7.54077
	30	24.7006	12.0151	7.63218	7.54072
	40	24.6934	12.0147	7.63216	7.54071
	50	24.6909	12.0146	7.63216	7.54071
	70	24.6892	12.0145	7.63215	7.54070
	80	24.6888	12.0145	7.63215	7.54070
	90	24.6887	12.0145	7.63215	7.54070
	100	24.6885	12.0145	7.63215	7.54070
$n = 2$	10	23.7099	11.4739	7.35940	7.27826
	20	23.4274	11.4633	7.35897	7.27795
	30	23.4038	11.4622	7.35892	7.27791
	40	23.3982	11.4620	7.35891	7.27790
	50	23.3962	11.4619	7.35891	7.27790
	70	23.3948	11.4618	7.35890	7.27790
	80	23.3946	11.4618	7.35890	7.27790
	90	23.3944	11.4618	7.35890	7.27790
	100	23.3943	11.4618	7.35890	7.27790
$n = 10$	10	22.4064	10.9570	7.09595	7.02410
	20	22.2373	10.9482	7.09588	7.02415
	30	22.2178	10.9473	7.09587	7.02415
	40	22.2134	10.9471	7.09587	7.02415
	50	22.2118	10.9470	7.09586	7.02415
	70	22.2107	10.9469	7.09586	7.02415
	80	22.2105	10.9469	7.09586	7.02415
	90	22.2104	10.9469	7.09586	7.02415
	100	22.2103	10.9469	7.09586	7.02415
$n = 50$	10	22.1354	10.8440	7.03737	6.96749
	20	21.9796	10.8355	7.03741	6.96766
	30	21.9609	10.8346	7.03742	6.96768
	40	21.9566	10.8344	7.03742	6.96768
	50	21.9551	10.8344	7.03742	6.96769
	70	21.9541	10.8343	7.03742	6.96769
	80	21.9539	10.8343	7.03742	6.96769
	90	21.9538	10.8343	7.03742	6.96769
	100	21.9537	10.8343	7.03742	6.96769

Table 4: Nusselt number for GITT with Bingham fluids.

	$n_{\max}$	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
$\eta_0 = 0$	10	25.2425	12.0299	7.63289	7.54128
	20	24.7301	12.0165	7.63224	7.54077
	30	24.7006	12.0151	7.63218	7.54072
	40	24.6934	12.0147	7.63216	7.54071
	50	24.6909	12.0146	7.63216	7.54071
	60	24.6898	12.0146	7.63215	7.5407
	70	24.6892	12.0145	7.63215	7.5407
	80	24.6888	12.0145	7.63215	7.5407
	90	24.6887	12.0145	7.63215	7.5407
	100	24.6885	12.0145	7.63215	7.5407
$\eta_0 = 0.25$	10	26.9423	12.6474	7.97251	7.86944
	20	26.0843	12.6305	7.97162	7.86873
	30	26.0475	12.6288	7.97154	7.86866
	40	26.0385	12.6283	7.97152	7.86865
	50	26.0353	12.6282	7.97151	7.86864
	60	26.0339	12.6281	7.9715	7.86864
	70	26.0332	12.6281	7.9715	7.86864
	80	26.0328	12.6281	7.9715	7.86864
	90	26.0325	12.6281	7.9715	7.86864
	100	26.0324	12.6281	7.9715	7.86864
$\eta_0 = 0.5$	10	30.4416	13.8057	8.51789	8.38528
	20	28.6656	13.7804	8.51659	8.38428
	30	28.6106	13.7778	8.51646	8.38418
	40	28.5972	13.7772	8.51643	8.38416
	50	28.5924	13.777	8.51642	8.38415
	60	28.5903	13.7769	8.51641	8.38414
	70	28.5892	13.7768	8.51641	8.38414
	80	28.5886	13.7768	8.51641	8.38414
	90	28.5882	13.7768	8.51641	8.38414
	100	28.588	13.7768	8.5164	8.38414
$\eta_0 = 0.75$	10	38.4557	16.0866	9.24541	9.07194
	20	34.4457	16.0331	9.24311	9.07025
	30	34.3367	16.0273	9.24283	9.07004
	40	34.307	16.0258	9.24275	9.06999
	50	34.2963	16.0252	9.24273	9.06997
	60	34.2916	16.025	9.24272	9.06996
	70	34.2892	16.0249	9.24271	9.06996
	80	34.2878	16.0248	9.24271	9.06995
	90	34.287	16.0248	9.24271	9.06995
	100	34.2865	16.0248	9.2427	9.06995

Table 5: Estimated error for FVM with different fluids.

	I	estimated relative error $\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
$\eta_0 = 0$	25	3.40E-02	4.80E-03	1.42E-03	1.74E-03
	50	5.24E-03	5.83E-04	2.45E-04	3.25E-04
	100	6.97E-04	8.32E-05	5.24E-05	7.43E-05
	200	8.91E-05	8.32E-06	1.05E-05	1.59E-05
	400	1.22E-05	0.00E+00	2.62E-06	3.98E-06
	800	0.00E+00	0.00E+00	1.31E-06	1.33E-06
	1600	0.00E+00	0.00E+00	0.00E+00	0.00E+00
	3200	0.00E+00	0.00E+00	0.00E+00	0.00E+00
$\eta_0 = 0.5$	25	2.47E-02	6.80E-03	1.66E-03	2.00E-03
	50	7.79E-03	8.86E-04	2.71E-04	3.58E-04
	100	1.08E-03	1.16E-04	5.28E-05	7.75E-05
	200	1.36E-04	1.45E-05	1.17E-05	1.67E-05
	400	1.75E-05	0.00E+00	2.35E-06	3.58E-06
	800	0.00E+00	0.00E+00	0.00E+00	1.19E-06
	1600	3.50E-06	0.00E+00	0.00E+00	1.19E-06
	3200	0.00E+00	0.00E+00	0.00E+00	0.00E+00
$n = 0.5$	25	3.3E-02	5.7E-03	1.5E-03	1.9E-03
	50	6.6E-03	7.4E-04	2.6E-04	3.4E-04
	100	9.0E-04	9.3E-05	5.3E-05	7.6E-05
	200	1.2E-04	1.5E-05	1.1E-05	1.6E-05
	400	1.5E-05	0.0E+00	2.5E-06	3.8E-06
	800	0.0E+00	0.0E+00	1.2E-06	1.3E-06
	1600	0.0E+00	0.0E+00	0.0E+00	0.0E+00
$n = 2$	25	3.1E-02	4.2E-03	1.4E-03	1.7E-03
	50	4.5E-03	5.0E-04	2.4E-04	3.2E-04
	100	5.9E-04	7.0E-05	5.2E-05	7.1E-05
	200	7.3E-05	8.7E-06	1.1E-05	1.8E-05
	400	8.5E-06	8.7E-06	4.1E-06	4.1E-06
	800	4.3E-06	0.0E+00	0.0E+00	1.4E-06
	1600	0.0E+00	0.0E+00	0.0E+00	0.0E+00

Table 6: Estimated error for GITT with different fluids.

	I	estimated relative error $\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
$\eta_0 = 0$	20	2.03E-02	1.11E-03	8.52E-05	6.76E-05
	30	1.19E-03	1.17E-04	7.86E-06	6.63E-06
	40	2.91E-04	3.33E-05	2.62E-06	1.33E-06
	50	1.01E-04	8.32E-06	0.00E+00	0.00E+00
	60	4.46E-05	0.00E+00	1.31E-06	1.33E-06
	70	2.43E-05	8.32E-06	0.00E+00	0.00E+00
	80	1.62E-05	0.00E+00	0.00E+00	0.00E+00
	90	4.05E-06	0.00E+00	0.00E+00	0.00E+00
	100	8.10E-06	0.00E+00	0.00E+00	0.00E+00
	110	4.05E-06	0.00E+00	0.00E+00	0.00E+00
	120	0.00E+00	0.00E+00	0.00E+00	0.00E+00
$\eta_0 = 0.5$	20	5.83E-02	1.83E-03	1.53E-04	1.19E-04
	30	1.92E-03	1.89E-04	1.53E-05	1.19E-05
	40	4.68E-04	4.35E-05	3.52E-06	2.39E-06
	50	1.68E-04	1.45E-05	1.17E-06	1.19E-06
	60	7.34E-05	7.26E-06	1.17E-06	1.19E-06
	70	3.85E-05	7.26E-06	0.00E+00	0.00E+00
	80	2.10E-05	0.00E+00	0.00E+00	0.00E+00
	90	1.40E-05	0.00E+00	0.00E+00	0.00E+00
	100	7.00E-06	0.00E+00	1.17E-06	0.00E+00
	110	7.00E-06	7.26E-06	0.00E+00	0.00E+00
	120	3.50E-06	0.00E+00	0.00E+00	0.00E+00
$n = 0.5$	20	4.2E-02	1.5E-03	1.2E-04	9.4E-05
	30	1.6E-03	1.5E-04	1.2E-05	8.8E-06
	40	3.9E-04	3.9E-05	3.7E-06	2.5E-06
	50	1.4E-04	1.5E-05	0.0E+00	1.3E-06
	70	9.7E-05	7.7E-06	1.2E-06	0.0E+00
	80	1.5E-05	7.7E-06	0.0E+00	0.0E+00
	90	1.1E-05	0.0E+00	0.0E+00	1.3E-06
	100	7.5E-06	0.0E+00	0.0E+00	0.0E+00
$n = 2$	20	1.2E-02	9.2E-04	5.8E-05	4.3E-05
	30	1.0E-03	9.6E-05	6.8E-06	5.5E-06
	40	2.4E-04	1.7E-05	1.4E-06	1.4E-06
	50	8.5E-05	8.7E-06	0.0E+00	0.0E+00
	70	6.0E-05	8.7E-06	1.4E-06	0.0E+00
	80	8.5E-06	0.0E+00	0.0E+00	0.0E+00
	90	8.5E-06	0.0E+00	0.0E+00	0.0E+00
	100	4.3E-06	0.0E+00	0.0E+00	0.0E+00

4 Conclusions

This paper presented a comparison between two solution strategies for calculating the Nusselt number in thermally developing channel flow: the Generalized Integral Transform Technique and the Finite Volumes Method. In order to properly compare both approaches, the original PDE problem was transformed into a ODE system, either through integral transformation (GITT) or discretization (FVM). Then the resulting ODE systems were solved using the same method. For this stage, both numerical solutions (using a numerical ODE system solver) or analytical solutions (using matrix exponentials) were employed.

The results, showed that, in general, the Finite Volumes solutions needed a very refined mesh to achieve six-digits convergence, while the GITT ones only needed a few terms of the sum, except near entry region. As suggestions for futures works, one should compare these techniques solving problems in others geometries like a tube or a complex geometry channel. In addition, a hybrid solution using both methodologies could be attempted, combining the good characteristics of each methodology.

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